# 1. PLANIMETRIJA

Planimetrija = geometrija ravnine, $M$ = Euklidska ravnina

## 1.1. Izometrije ravnine

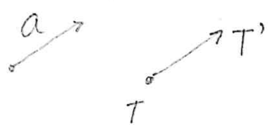
def: Za preslikavanje $f: M \to M$ kažemo da je **izometrija** ako „čuva udaljenost", tj. ako za bilo koji par točaka $A, B \in M$ vrijedi $|f(A)f(B)| = |AB|$.

Primjeri: ① Identiteta $id: M \to M$, $id(T) = T \ \forall T \in M$,

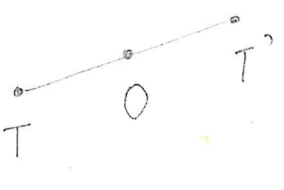
② Translacija za vektor $\vec{a}$, $t_{\vec{a}}: M \to M$

$t_{\vec{a}}(T) = T'$

gdje je $\overrightarrow{TT'} = \vec{a}$



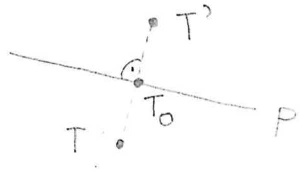

③ Centralna simetrija obzirom na točku $O$, $s_O: M \to M$,
$s_O(T) = T'$ t.d. je $O$ polovište dužine $\overline{TT'}$.




④ Osna simetrija obzirom na pravac $p$, $s_p: M \to M$,
$s_p(T) = T'$ t.d.

1.) $T = T'$ ako je $T \in p$

2.) $TT' \perp p$ i $|TT_0| = |T_0T'|$ za $T_0 = p \cap TT'$



⑤ Rotacija oko točke $O$ za kut $\alpha$, $r_\alpha : M \to M$, $r_\alpha(T) = T'$ t.d. $|OT| = |OT'|$ i $\sphericalangle TOT' = \alpha$

Napomena: Za pozitivan kut podrazumijevamo rotaciju u smjeru suprotnom od kazaljke na satu:

$\alpha > 0$ ↺ $\qquad\qquad$ $\alpha < 0$ ↻

Sve ostale izometrije dobivaju se kompozicijom navedenih.

Zad 1.: Dokažite da su izometrije injekcije.

Rj: $f(A) = f(B) \implies 0 = |f(A)f(B)| = |AB| \implies A = B$

DZ: Dokažite da je kompozicija dviju izometrija također izometrija. Dokažite da je inverzno preslikavanje izometrije također izometrija.

Zad 2.: Odredite preslikavanje inverzno translaciji $t_{\vec{a}}$, centralnoj simetriji $s_O$, osnoj simetriji $s_p$ i rotaciji $r_\alpha$.

Rj: $(t_{\vec{a}})^{-1} = t_{-\vec{a}}$ (translacija za suprotni vektor)

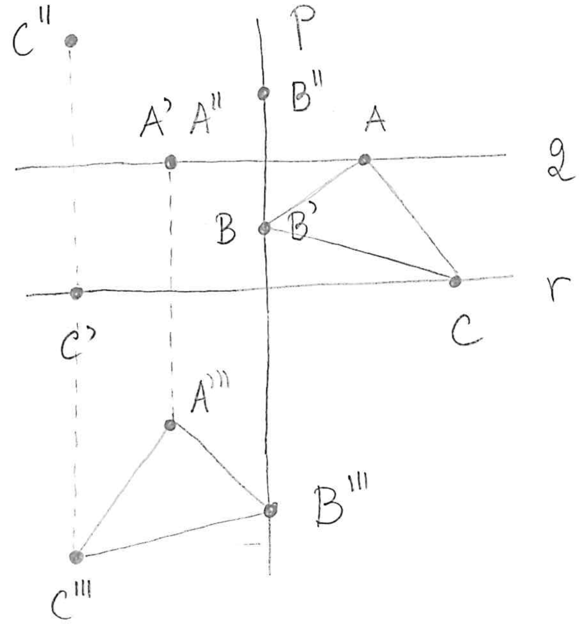
$s_O^{-1} = s_O$, $\quad s_p^{-1} = s_p$ (simetrije su same sebi inverzne).

Napomena: Preslikavanja s tim svojstvom $(f = f^{-1})$ se zovu involucije

$(r_\alpha)^{-1} = r_{-\alpha}$ (rotacija oko iste točke za suprotni kut)

**Zad 3:** Dani su pravci $p, q$ i $r$ t.d. $q \parallel r$, $p \perp q$ i točke $A \in q$, $B \in p$, $C \in r$ t.d. $d(B,q) = d(B,r)$. Odredite sliku trokuta $ABC$ pri izometriji $f = s_r \circ s_q \circ s_p$.

**Rj:**



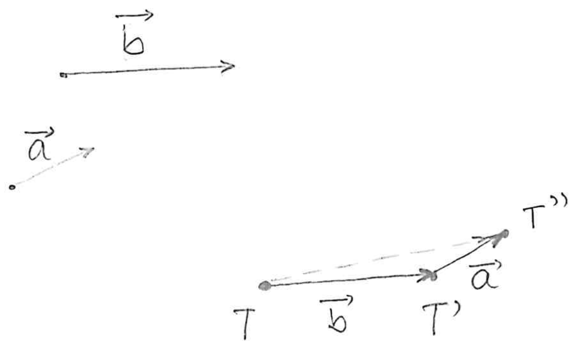

$s_p : A, B, C \longmapsto A', B', C'$

$s_q : A', B', C' \longmapsto A'', B'', C''$

$s_r : A'', B'', C'' \longmapsto A''', B''', C'''$

**Zad 4:** Što je kompozicija translacija $t_{\vec{a}} \circ t_{\vec{b}}$? Rotacija $r_\alpha \circ r_\beta$ oko iste točke $O$? Simetrija $s_p \circ s_q$?

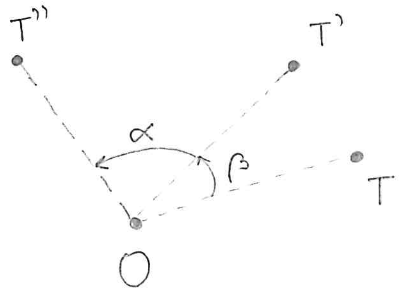
**Rj:**



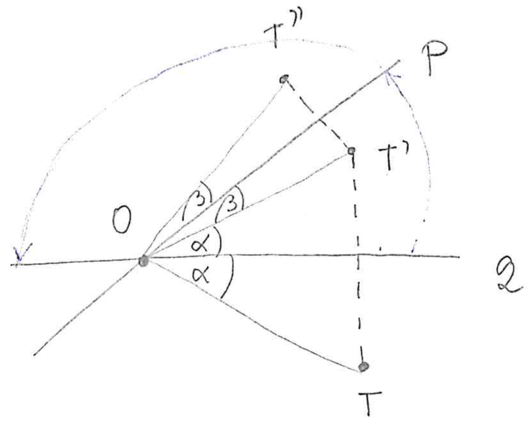

$T'' = t_{\vec{a}+\vec{b}}(T)$

Dakle, $t_{\vec{a}} \circ t_{\vec{b}} = t_{\vec{a}+\vec{b}}$

(iz toga vidimo da je komponiranje translacija komutativno).




$r_\alpha \circ r_\beta = r_{\alpha+\beta}$




$|OT| = |OT'| = |OT''|$

$\sphericalangle TOT'' = 2\alpha + 2\beta = 2 \sphericalangle(q, p)$

$s_p \circ s_q$ je rotacija oko sjecišta $O = p \cap q$ za dvostruki kut među pravcima $p, q$.

Je li svejedno u kojem redoslijedu komponiramo?

Ne, ako komponiramo u obrnutom redoslijedu, onda gledamo kut $180° - (\alpha + \beta)$.

## 1.2. Sukladnost trokuta

**def:** Za trokute $\triangle ABC$ i $\triangle A'B'C'$ kažemo da su sukladni ako se podudaraju u sve tri stranice i sva tri kuta, tj. $|AB| = |A'B'|$, $|AC| = |A'C'|$, $|BC| = |B'C'|$, $\sphericalangle BAC = \sphericalangle B'A'C'$, $\sphericalangle ABC = \sphericalangle A'B'C'$, $\sphericalangle ACB = \sphericalangle A'C'B'$.

**Alternativna definicija:** trokuti su sukladni ako postoji izometrija ravnine koja preslikava jednog u drugog.

Oznaka za sukladnost: $\triangle ABC \cong \triangle A'B'C'$

**Teoremi o sukladnosti:**

**Tm (SSS):** $a = a'$, $b = b'$, $c = c'$ $\Longrightarrow \triangle ABC \cong \triangle A'B'C'$

**Tm (SKS):** (dvije stranice i kut među njima)

$a = a'$, $b = b'$, $\gamma = \gamma'$ $\Longrightarrow \triangle ABC \cong \triangle A'B'C'$

**Teorem (KSK):** (stranica i dva kuta uz nju)

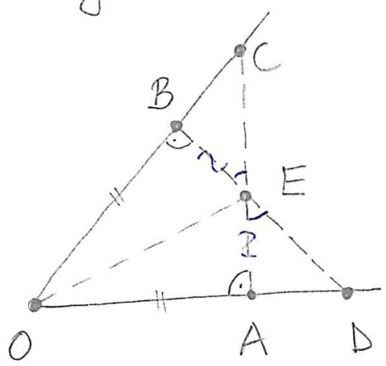
$$a=a', \ \beta=\beta', \ \gamma=\gamma' \Longrightarrow \Delta ABC \cong \Delta A'B'C'$$

**Teorem ($SSK^{>}$):** (dvije stranice i kut nasuprot većoj)

$$a>b, \ a=a', \ b=b', \ \alpha=\alpha' \Longrightarrow \Delta ABC \cong \Delta A'B'C'$$

**Zad 5.:** Na krakovima kuta s vrhom O dane su točke A i B tako da je $|OA|=|OB|$. Okomice na pripadajuće krakove u točkama A i B sijeku suprotni krak u točkama C i D, a međusobno se sijeku u točki E. Dokažite da je $\Delta BCE$ sukladan trokutu $\Delta ADE$

**Rj:**




Prema teoremu $SSK^{>}$ vrijedi

$\Delta OAE \cong \Delta OBE$

$(|OA|=|OB|, \ |OE|=|OE|,$
$\measuredangle OAE = \measuredangle OBE = 90^\circ)$

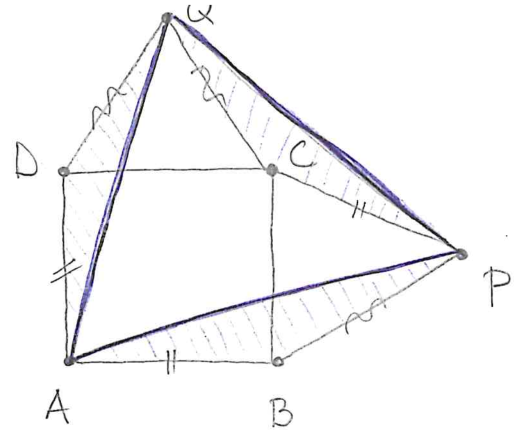
Slijedi $|BE|=|AE|$. Sada primjenimo teorem KSK:

$|AE|=|BE|, \quad \measuredangle EAD = \measuredangle EBC = 90^\circ, \quad \measuredangle AED = \measuredangle BEC$

(vršni kutevi). Dakle, $\Delta ADE \cong \Delta BCE$.

**Zad 6.:** Nad stranicama $\overline{BC}$ i $\overline{CD}$ kvadrata ABCD konstruirani su jednakostranični trokuti $\Delta BPC$ i $\Delta DCQ$. Dokažite da je trokut $\Delta APQ$ jednakostraničan.

Rj:




Treba dokazati $|AP|=|PQ|=|QA|$. Dokazat ćemo da su trokuti $\triangle ABP$, $\triangle ADQ$ i $\triangle PCQ$ sukladni.

$|AB|=|AD|=|PC|$

$|BP|=|DQ|=|CQ|$

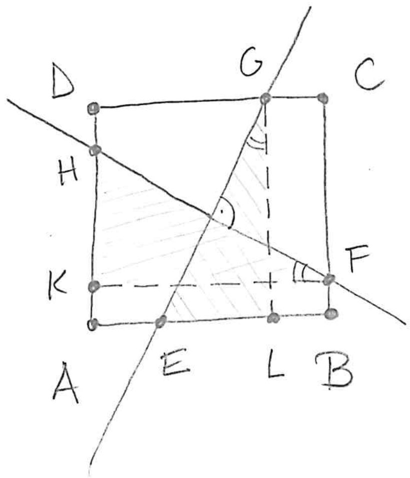
$\measuredangle ABP = \measuredangle ADQ = 90^\circ + 60^\circ = 150^\circ$, $\measuredangle PCQ = 360^\circ - 90^\circ - 60^\circ - 60^\circ = 150^\circ$

Po teoremu SKS $\Rightarrow \triangle ABP \cong \triangle ADQ \cong \triangle PCQ$

$\Rightarrow |AP|=|PQ|=|QA|$.

zad 7.: Dva okomita pravca siječu stranice $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ kvadrata $ABCD$ redom u točkama $E, F, G, H$. Dokažite da je $|EG|=|HF|$.

Rj:




Nadopunimo sliku točkama $K \in \overline{AD}$ i $L \in \overline{AB}$ t.d. je $FK \parallel AB$ i $GL \parallel BC$.

$|GL|=|FK|$ (duljina stranice kvadrata)

$\measuredangle ELG = \measuredangle HKF = 90^\circ$

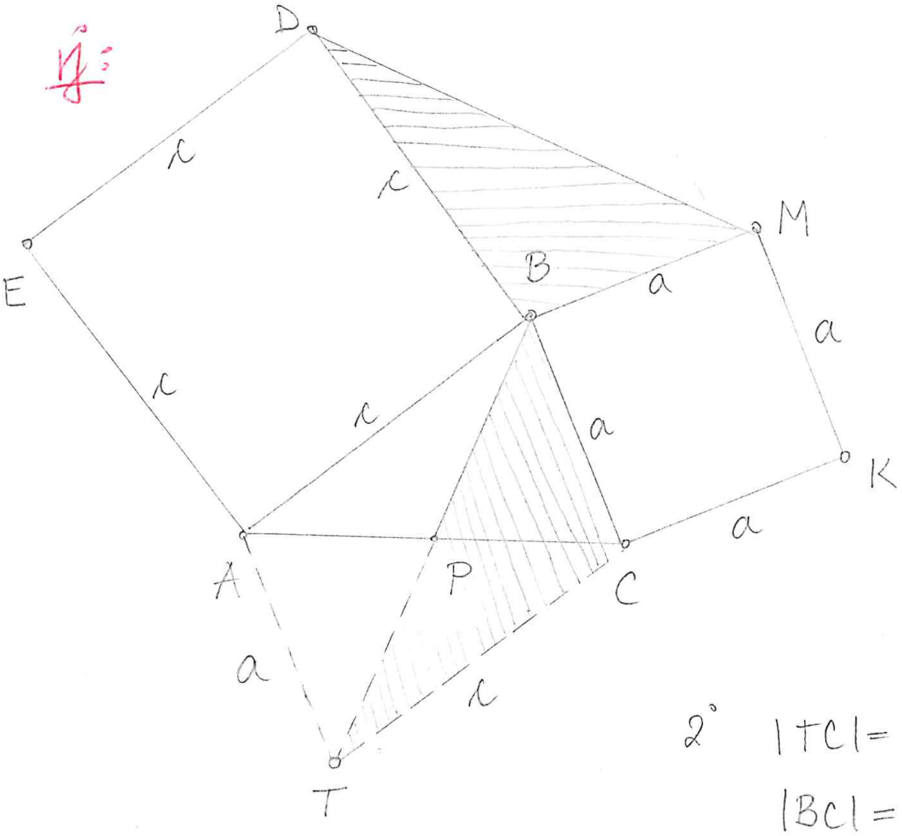
$\measuredangle LGE = \measuredangle KFH$ (kutovi s okomitim kracima)

Po teoremu KSK je $\triangle GLE \cong \triangle FKH$.

Slijedi $|EG|=|HF|$.

**Zad 8:** Nad stranicama $\overline{AB}$ i $\overline{BC}$ trokuta $\triangle ABC$ konstruirani su kvadrati $ABDE$ i $BCKM$. Dokažite da je duljina dužine $\overline{DM}$ dva puta veća od duljine težišnice $\overline{BP}$ trokuta $\triangle ABC$.

**Rj:**




1° Neka je $T$ točka na pravcu $BP$ različita od $T$ t.d. $|\overline{BP}|=|\overline{TP}|$.

$\triangle APB \cong \triangle CPT$ jer je $|AP|=|PC|$, $|BP|=|PT|$ i $\measuredangle APB = \measuredangle TPC$ (vršni kutovi). Stoga je $|TC|=|AB|=c$

2° $|TC|=|BD|=c$

$|BC|=|BM|=a$

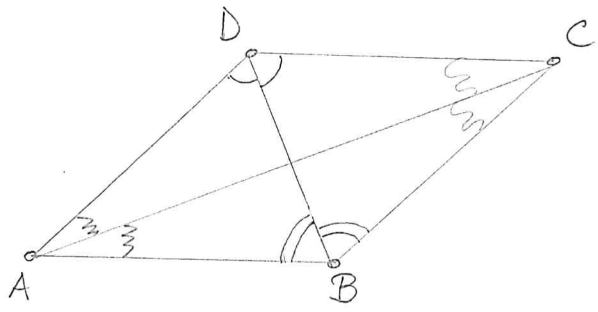
Četverokut $ABCT$ je paralelogram $\Rightarrow$ $\left.\begin{array}{l} TC \parallel AB \\ BD \perp AB \end{array}\right\} \Rightarrow \left.\begin{array}{l} TC \perp BD \\ CB \perp BM \end{array}\right\} \Rightarrow \measuredangle BCT = \measuredangle MBD$ (kuteri s okomitim kracima)

Po S-K-S teoremu $\Rightarrow \triangle BCT \cong \triangle MBD$ pa je $|\overline{DM}|=|\overline{BT}|$

$\Rightarrow |\overline{DM}| = 2\cdot|\overline{BP}|$.

**Zad 9.:** U četverokutu $ABCD$ dijagonale raspolavljaju kutere. Dokažite da je taj četverokut romb, tj. da su sve četiri stranice jednake duljine.

Rj:




$\sphericalangle ADB = \sphericalangle BDC$
$\sphericalangle ABD = \sphericalangle CBD$
$|\overline{BD}| = |\overline{BD}|$

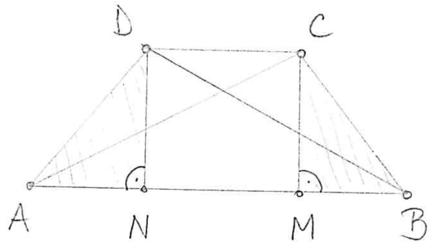
K-S-K $\Downarrow$ $\triangle ABD \cong \triangle CBD$

$\Longrightarrow |AB| = |BC|$ & $|\overline{AD}| = |\overline{DC}|$

Analogno dobijemo $\triangle ACB \cong \triangle ACD$ pa je $|\overline{AD}| = |\overline{AB}|$ & $|\overline{BC}| = |\overline{CD}|$. Dakle, $|\overline{AB}| = |\overline{BC}| = |\overline{CD}| = |\overline{DA}|$.

Zad 10.: Dokažite da je trapez kojemu su dijagonale jednake jednakokračan.

Rj:




Znamo $|\overline{AC}| = |\overline{BD}|$. Treba dokazati $|\overline{AD}| = |\overline{BC}|$.

Neka su M i N nožišta visina iz vrhova C i D na AB.

Tada vrijedi

$|\overline{DN}| = |\overline{CM}|$ = visina trapeza
$|\overline{AC}| = |\overline{BD}|$
$\sphericalangle AMC = \sphericalangle BND = 90°$

$SSK^{>}$ $\Downarrow$ $\triangle AMC \cong \triangle BND$

$\Longrightarrow |\overline{AM}| = |\overline{BN}|$

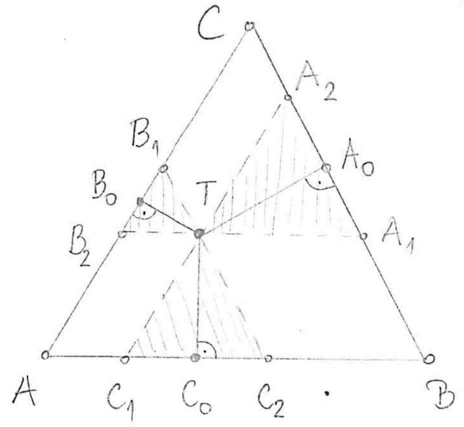
Sada je

$|AN| = |AB| - |NB| = |AB| - |AM| = |BM|$
$|DN| = |CM|$
$\sphericalangle AND = \sphericalangle BMC = 90°$

SKS $\Downarrow$ $\triangle AND \cong \triangle BMC$ $\Downarrow$ $|AD| = |BC|$.

**zad 11.:** Dokažite da je zbroj udaljenosti bilo koje točke unutar jednakostraničnog trokuta do njegovih stranica konstantan i jednak duljini visine trokuta.

**rj:**




Povucimo kroz T paralele sa stranicama trokuta i označimo sjecišta kao na slici.

$\sphericalangle B_1B_2T = \sphericalangle B_2AC_1 = 60°$ (kutevi s paralelnim kracima)

Analogno se pokaže $\sphericalangle B_2B_1T = 60°$, $\sphericalangle TA_2A_1 = \sphericalangle TA_1A_2 = \sphericalangle TC_1C_2 = \sphericalangle C_1C_2T = 60°$. Stoga su trokuti $\Delta B_1TB_2$, $\Delta A_1TA_2$ i $\Delta C_1TC_2$ jednakostranični, a $\overline{TA_0}, \overline{TB_0}, \overline{TC_0}$ su njihove visine. Stoga je

$$|TA_0|+|TB_0|+|TC_0| = \frac{\sqrt{3}}{2}\left(|TA_1|+|TB_2|+|C_1C_2|\right)$$

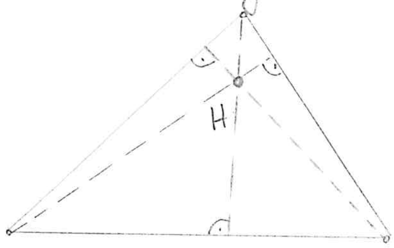
Kako su četverokuti $AC_1TB_2$, $BA_1TC_2$, $CB_1TA_2$ paralelogrami, to je $|TA_1| = |BC_2|$, $|TB_2| = |C_1A|$ pa je

$$|TA_0|+|TB_0|+|TC_0| = \frac{\sqrt{3}}{2}\left(|BC_2|+|C_1A|+|C_1C_2|\right) = \frac{\sqrt{3}}{2}|AB| =$$

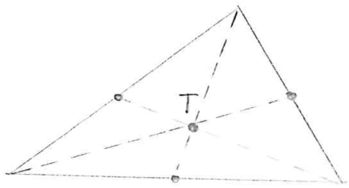
duljina visine trokuta $\Delta ABC$.

# 1.3. Karakteristične točke trokuta

1. Ortocentar H – sjecište visina trokuta

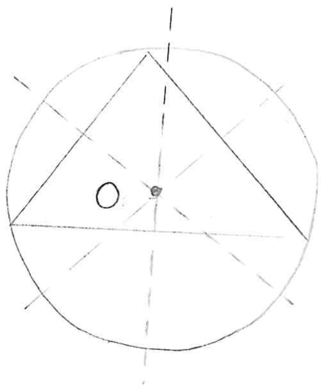



2. Težište T – sjecište težišnica trokuta




Težište dijeli svaku težišnicu u omjeru 1:2.

3. Sjecište simetrala stranica O

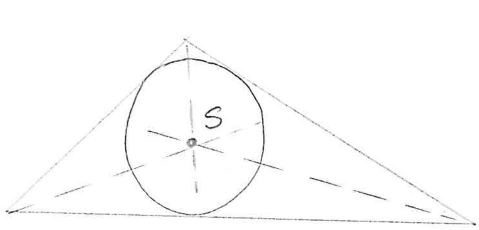



Točka O je ujedno središte trokutu opisane kružnice. Naime, simetrale stranica su jednako udaljene od dva vrha, a točka O je jednako udaljena od sva tri vrha.

Polumjer opisane kružnice $R = \frac{abc}{4P}$, gdje je P površina trokuta. Površinu možemo izračunati preko Heronove formule $P = \sqrt{s(s-a)(s-b)(s-c)}$, gdje je

$s = \frac{a+b+c}{2}$ = poluopseg.

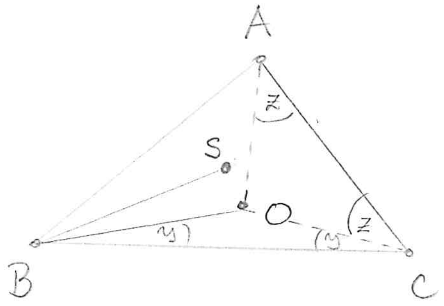
4. Sjecište simetrala kutova S




To je ujedno središte trokutu upisane kružnice.
Polumjer upisane kružnice je $r = \frac{P}{s}$
($s$ = poluopseg)

**Zad 12.:** Točke $O$ i $S$ su redom središta opisane i upisane kružnice trokuta. Izrazite kut $\measuredangle OBS$ pomoću kutova trokuta $\alpha, \beta, \gamma$.

**Rj:**




Pravac $BS$ je simetrala kuta $\measuredangle ABC$ pa je $\measuredangle SBA = \frac{\beta}{2}$.

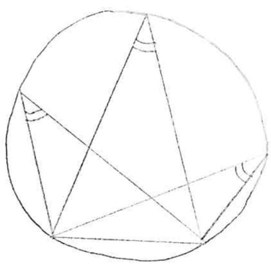
Kako je $|OA| = |OB| = |OC|$, to su trokuti $\triangle OAB$, $\triangle OBC$ i $\triangle OAC$ jednakokračni pa je $\measuredangle OAB = \measuredangle OBA = x$, $\measuredangle OBC = \measuredangle OCB = y$, $\measuredangle OAC = \measuredangle OCA = z$. Vidimo da je

$$\left.\begin{array}{l} x+y=\beta \\ x+z=\alpha \\ y+z=\gamma \end{array}\right\} \Rightarrow \left.\begin{array}{l} x+y=\beta \\ x-y=\alpha-\gamma \end{array}\right\} \Rightarrow x = \frac{\alpha+\beta-\gamma}{2}$$

$$\Rightarrow \measuredangle OBA = \frac{\alpha+\beta-\gamma}{2}$$

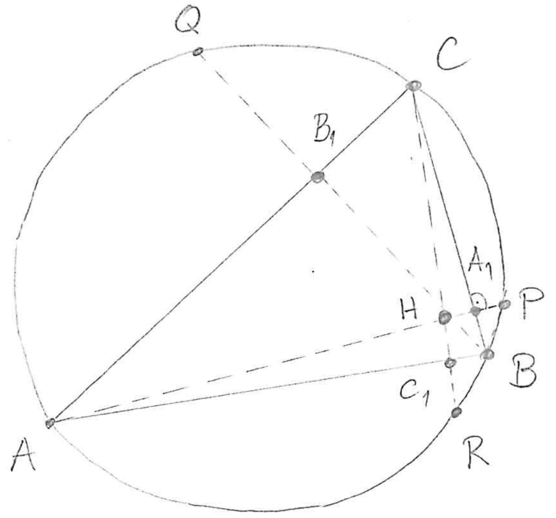
$$\measuredangle OBS = |\measuredangle OBA - \measuredangle SBA| = \left|\frac{\alpha+\beta-\gamma}{2} - \frac{\beta}{2}\right| = \left|\frac{\alpha-\gamma}{2}\right|$$

**Teorem:** Obodni kutovi nad istim lukom kružnice su jednaki.



**Zad 13.:** Dokažite da točke simetrične ortocentru obzirom na stranice trokuta leže na opisanoj kružnici.

Rj:




Neka su $P, Q$ i $R$ sjecišta pravaca $AH$, $BH$ i $CH$ s opisanom kružnicom (tim redoslijedom). Treba pokazati $|HA_1|=|A_1P|$, $|HB_1|=|B_1Q|$, $|HC_1|=|C_1R|$

Dokažimo $|HA_1|=|A_1P|$ (ostale dvije tvrdnje se analogno dokazuju).

$$\sphericalangle A_1BP = \sphericalangle CBP \underset{\substack{\uparrow \\ \text{nad istim} \\ \text{lukom}}}{=} \sphericalangle CAP = \sphericalangle CAA_1 = 90^\circ - \gamma$$

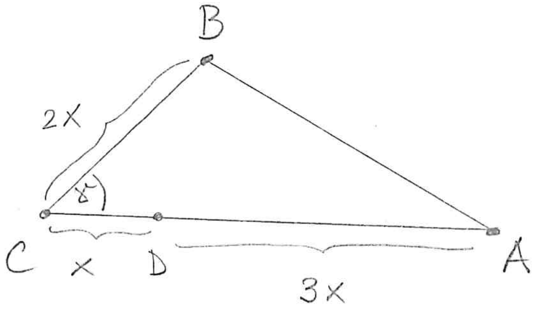
$$\sphericalangle A_1BH = \sphericalangle CBB_1 = 90^\circ - \gamma$$

Dakle,
$$\left.\begin{aligned} \sphericalangle A_1BP &= \sphericalangle A_1BH \\ \sphericalangle HA_1B &= \sphericalangle PA_1B = 90^\circ \\ \overline{A_1B} &= \overline{A_1B} \end{aligned}\right\} \overset{K-S-K}{\Downarrow} \quad \Delta HA_1B \cong \Delta PA_1B \Rightarrow |HA_1| = |A_1P|.$$

~~Zad~~ 14.:

U trokutu $\Delta ABC$ vrijedi $b=2a$ i $\cos\gamma = \frac{3}{4}$. Na stranici $\overline{AC}$ nalazi se točka $D$ i dijeli je u omjeru $|AD|:|DC|=3:1$. Odredite omjer polumjera kružnice opisane trokutu $\Delta ABC$ i kružnice upisane trokutu $\Delta ABD$.

rj:




Označimo $|CD|=x$. Onda je

$|AD|=3x \Rightarrow |AC|=4x=b=2a$

$\Rightarrow a=2x$

Iz kosinusovog poučka dobivamo $c^2=a^2+b^2-2ab\cos\gamma=$

$=4x^2+16x^2-2\cdot 2x\cdot 4x\cdot\frac{3}{\cancel{4}}=20x^2-12x^2=8x^2 \Rightarrow \boxed{c=2\sqrt{2}x}$

$R=\frac{abc}{4P}$ $\quad P=\frac{1}{2}|AC|\cdot|BC|\cdot\sin\gamma=\frac{1}{\cancel{2}}\cdot 4x\cdot\cancel{2}x\cdot\sqrt{1-\frac{9}{16}}=$

$=x^2\cdot\sqrt{7}$

$$\boxed{R=\frac{16\sqrt{2}x^3}{4\sqrt{7}x^2}=\frac{4\sqrt{2}}{\sqrt{7}}x}$$

Nadalje, vrijedi $r=\frac{P}{s}$

$|BD|^2=x^2+4x^2-2\cdot 2x\cdot x\cdot\frac{3}{4}=2x^2 \Rightarrow |BD|=\sqrt{2}x$

$$s=\frac{|BD|+|DA|+|AB|}{2}=\frac{\sqrt{2}x+3x+2\sqrt{2}x}{2}=\frac{3(1+\sqrt{2})x}{2}$$

$$P_{\triangle ABD}=\frac{3}{4}P_{\triangle ABC}=\frac{3\sqrt{7}x^2}{4} \Longrightarrow r=\frac{\frac{3\sqrt{7}}{4}x^2}{\frac{3(1+\sqrt{2})}{2}x}=\frac{\sqrt{7}x}{2(1+\sqrt{2})}$$

$\uparrow$

jer imaju iste visine, a stranica je za faktor $\frac{3}{4}$ manja

Traženi omjer je $$\frac{R}{r}=\frac{\frac{4\sqrt{2}x}{\sqrt{7}}}{\frac{\sqrt{7}x}{2(1+\sqrt{2})}}=\frac{8\sqrt{2}(1+\sqrt{2})}{7}=\frac{8(\sqrt{2}+2)}{7}$$

# 1.4. SLIČNOST

def: Za trokute $\Delta$, $\Delta'$ kažemo da su <u>slični</u> i pišemo $\Delta \sim \Delta'$ ako je $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$ i $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

## Teoremi o sličnosti:

Teorem (KKK) $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$ $\Longrightarrow \Delta \sim \Delta'$

Teorem (SSS) $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} \Longrightarrow \Delta \sim \Delta'$

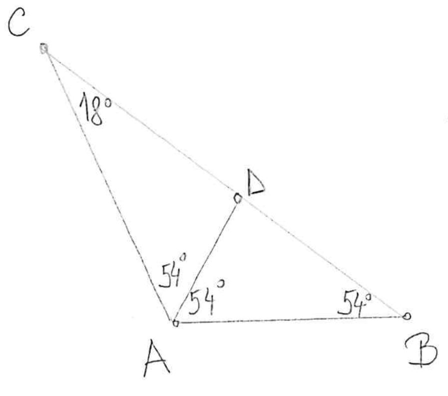
Teorem (SKS) $\frac{a}{a'} = \frac{b}{b'}$ i $\gamma = \gamma' \Longrightarrow \Delta \sim \Delta'$

Teorem (SSK$^>$) $a > b$, $\frac{a}{a'} = \frac{b}{b'}$, $\alpha = \alpha' \Longrightarrow \Delta \sim \Delta'$

zad 15.: Kutevi trokuta odnose se kao 6:3:1. Dokažite da simetrala najvećeg kuta odsijeca od danog trokuta njemu sličan trokut.

rj: $\alpha = 6\gamma$, $\beta = 3\gamma$, $\alpha + \beta + \gamma = 180° \Longrightarrow 10\gamma = 180° \Longrightarrow \boxed{\gamma = 18°}$

$\Longrightarrow \boxed{\alpha = 108°, \beta = 54°}$

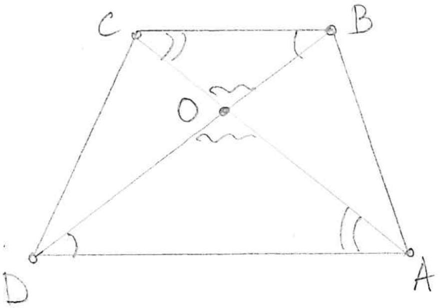



$\sphericalangle CAD = \frac{108°}{2} = 54° \Longrightarrow \sphericalangle CDA = 180° - 18° - 54° = 108°$

Po teoremu KKK $\Longrightarrow \Delta ABC \sim \Delta DAC$.

zad 16.: Dijagonale četverokuta ABCD sijeku se u točki O. Dokažite $|AO| \cdot |BO| = |CO| \cdot |DO| \Longleftrightarrow AD \parallel BC$

Rj: $\boxed{\Longleftarrow}$ Pretpostavimo da je $AD \parallel BC$.




$\sphericalangle ODA = \sphericalangle OBC$ } kuteri sa uzajamno
$\sphericalangle OAD = \sphericalangle OCB$ } paralelnim kracima

$\sphericalangle DOA = \sphericalangle COB$ vršni kuteri

Po teoremu KKK vrijedi $\triangle AOD \sim \triangle COB$ pa je

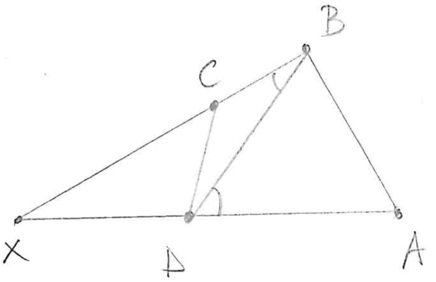
$$\frac{|AO|}{|CO|} = \frac{|DO|}{|BO|} \Longrightarrow |AO|\cdot|BO| = |CO|\cdot|DO|.$$

$\boxed{\Longrightarrow}$ Neka vrijedi $|AO|\cdot|BO|=|CO|\cdot|DO| \Longrightarrow \dfrac{|AO|}{|CO|} = \dfrac{|DO|}{|BO|}$.

Zbog $\sphericalangle AOD = \sphericalangle COB$ po tm-u SKS $\Longrightarrow \triangle AOD \sim \triangle COB$.

$\Longrightarrow \sphericalangle ADO = \sphericalangle CBO$, tj. $\sphericalangle ADB = \sphericalangle CBD$, a onda je $AD \parallel BC$. Zašto?

Pretpostavimo suprotno, tj. $AD \nparallel BC$. Tada se $AD$ i $BC$ sijeku u točki $X$.

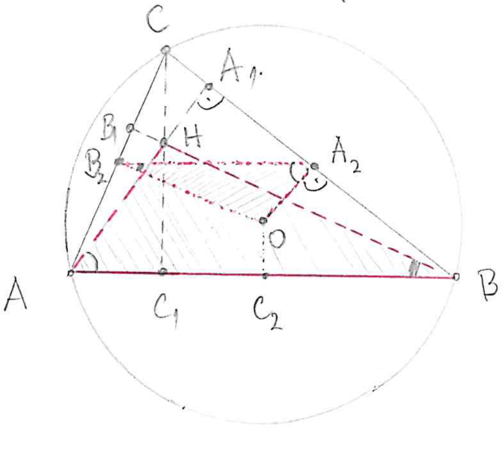



Stavimo $\sphericalangle ADB = \sphericalangle CBD = \omega$

Tada je $\sphericalangle XDB = 180° - \omega$ pa je $\sphericalangle BXD = 180° - \sphericalangle XBD - \sphericalangle XDB = 0°$ ↯

Zad 17.: Dan je trokut $\triangle ABC$. Dokažite da je udaljenost ortocentra od vrha $A$ dva puta veća od udaljenosti središta opisane kružnice od stranice $\overline{BC}$.

Rj:




Neka su $A_1, B_1, C_1$ nožišta visina iz $A$, $B$, odnosno $C$, a $A_2, B_2, C_2$ polovišta stranica $\overline{BC}$, $\overline{CA}$ i $\overline{AB}$ redom.

Sada je $AH \parallel OA_2$ (jer su oba pravca okomita na BC)

$BH \parallel OB_2$ (jer su oba pravca okomita na AC)

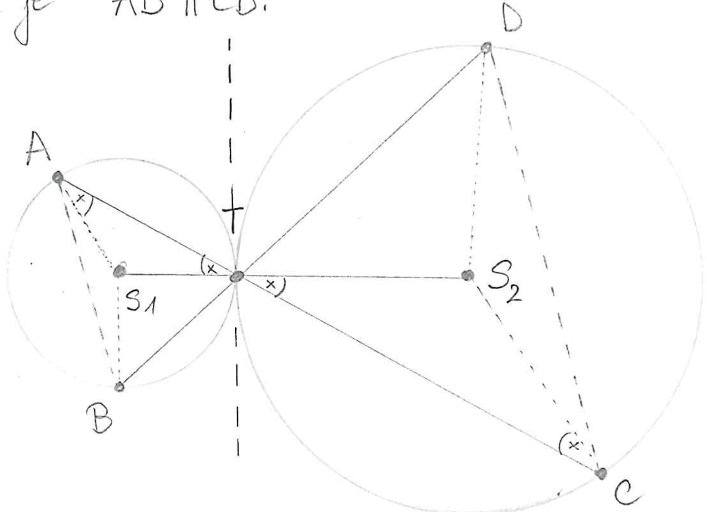
$AB \parallel A_2B_2$ (jer je $A_2B_2$ srednjica trokuta)

Stoga je
$$\left.\begin{aligned} \sphericalangle HAB &= \sphericalangle B_2A_2O \\ \sphericalangle HBA &= \sphericalangle A_2B_2O \\ \sphericalangle AHB &= \sphericalangle A_2OB_2 \end{aligned}\right\} \text{kutevi sa uzajamno paralelnim kracima}$$

Po teoremu KKK je $\Delta ABH \sim \Delta A_2B_2O$. Koeficijent sličnosti je $\frac{|AB|}{|A_2B_2|} = 2 \Longrightarrow \frac{|AH|}{|A_2O|} = 2 \Rightarrow |AH| = 2|A_2O|$ ■

**Zad 18.:** Dvije kružnice dodiruju se izvana u točki T. Kroz točku T prolaze dvije sekante koje prvu kružnicu sijeku u točkama A, B, a drugu u C, D. Dokažite da je $AB \parallel CD$.

**Rj:**




$$\left.\begin{aligned} &\sphericalangle ATS_1 = \sphericalangle CTS_2 = x \text{ (vršni kutevi)} \\ &\Delta ATS_1 \text{ i } \Delta CTS_2 \text{ su jednakokračni} \\ &\Rightarrow \sphericalangle TAS_1 = \sphericalangle ATS_1 = x \\ &\quad \sphericalangle CTS_2 = \sphericalangle TCS_2 = x. \end{aligned}\right\} \Rightarrow \Delta ATS_1 \sim \Delta CTS_2$$

Analogno se dokaže $\Delta BTS_1 \sim \Delta DTS_2$

Slijedi $\dfrac{|AT|}{|CT|} = \dfrac{|S_1T|}{|S_2T|} = \dfrac{|BT|}{|DT|}$

Nadalje, $\sphericalangle ATB = \sphericalangle CTD$ (vršni kuteni) $\Big\} \underset{(SKS)}{\Longrightarrow} \triangle ATB \sim \triangle CTD$

Posebno, $\sphericalangle ABT = \sphericalangle CDT$, tj. $\sphericalangle ABD = \sphericalangle CDB$, a onda je $AB \parallel CD$.

zadaća:

1. Unutar trokuta $\triangle ABC$ odabrana je točka $P$ tako da su kuteni $\sphericalangle PAC$ i $\sphericalangle PBC$ jednaki. Nožišta okomica iz $P$ na $AC$ i $BC$ su $M$ i $N$. Ako je $D$ polovište od $\overline{AB}$, dokažite da je $|DM| = |DN|$.

2. Kraćeni i kraća osnovica trapeza tangente su kružnice sa središtem na dužoj osnovici. Dokažite da je duljina duže osnovice jednaka zbroju duljina krakova.

3. Dokažite da polovišta stranica četverokuta tvore paralelogram.

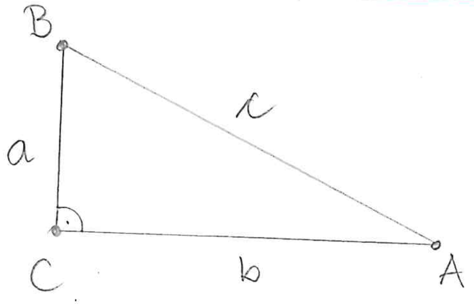
4. Dan je pravokutnik $ABCD$ t.d. vrijedi $|AD| = |AB|\sqrt{2}$. Točka $K$ je polovište stranice $\overline{AD}$. Odredite kut između pravaca $AC$ i $BK$.

5. Dokažite da središte upisane kružnice trokuta dijeli simetralu kuta u omjeru zbroja susjednih stranica prema trećoj stranici.

6. Dan je trokut sa stranicama $|AB| = 16$, $|AC| = 12$ i $|BC| = 20$. Na stranici $\overline{AB}$ nalaze se točke $M, P$, a na stranici $\overline{BC}$ točke $N, Q$ t.d. $MN \parallel PQ \parallel AC$ i $|CN| = |BQ| = 5$. Odredite duljine stranica trapeza $MPQN$.

7. Iz vrha A paralelograma ABCD spuštene su okomice AM i AN na pravce BC i CD. Dokažite da su trokuti $\Delta ABC$ i $\Delta AMN$ slični.

## 1.5. PITAGORIN POUČAK

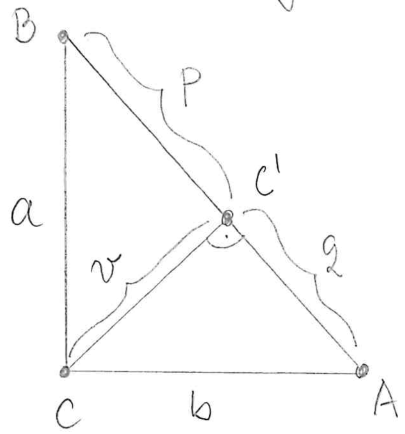



Ako je $\Delta ABC$ pravokutan s pravim kutem pri vrhu C, onda je $c^2=a^2+b^2$.

Obrat: Ako vrijedi $c^2=a^2+b^2$, onda je $\Delta ABC$ pravokutan s pravim kutem pri vrhu C.

Zad 19.: U pravokutnom trokutu visina dijeli hipotenuzu na dužine duljina p i q. Dokažite da je duljina visine $v=\sqrt{pq}$.

Rj:




$$a^2+b^2=c^2=(p+q)^2$$

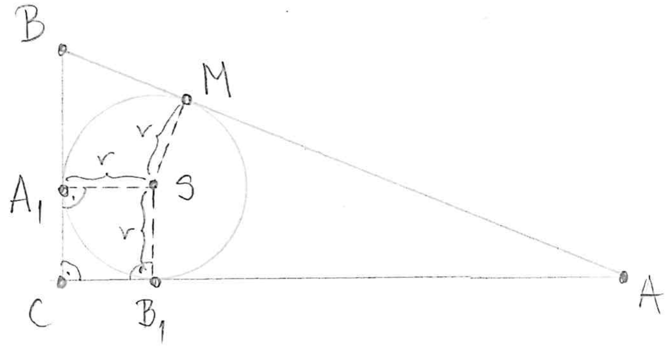
$$\left.\begin{aligned} a^2&=p^2+v^2 \\ b^2&=q^2+v^2 \end{aligned}\right\} \Rightarrow \cancel{p^2}+\cancel{q^2}+2v^2=a^2+b^2=$$

$$=\cancel{p^2}+2pq+\cancel{q^2}$$

$$\Rightarrow v^2=pq \Rightarrow v=\sqrt{pq}.$$

Zad 20.: Upisana kružnica dodiruje hipotenuzu pravokutnog trokuta u točki M. Dokažite da je površina trokuta jednaka $|AM|\cdot|BM|$.

Rj:




$SA_1CB_1$ je kvadrat (tri prava kuta)

$\Delta ASB_1 \cong \Delta ASM$ (SSK>)

$\Rightarrow |AM|=|AB_1|=|AC|-|CB_1|=b-r.$

Analogno dobijemo $|BM| = |BA_1| = a - r$

$$c = |AM| + |BM| = b - r + a - r \Rightarrow r = \frac{a+b-c}{2}$$

$$|AM| \cdot |BM| = (b-r)(a-r) = \frac{b-a+c}{2} \cdot \frac{a-b+c}{2} = \frac{1}{4}\left(c^2 - (a-b)^2\right)$$

$$= \frac{1}{4}\left(\cancel{a^2} + \cancel{b^2} - \cancel{a^2} - \cancel{b^2} + 2ab\right) = \frac{1}{2}ab = \text{površina trokuta.}$$

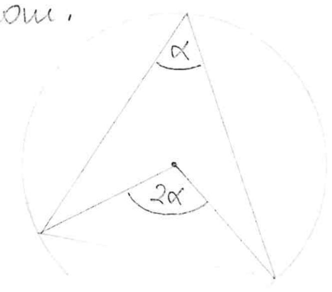
## 1.6. Kružnica

**def**: Kružnica sa središtem u $S$ je skup točaka u ravnini koje su jednako udaljene od točke $S$.

Koristimo oznake: $k_S(r)$ = kružnica polumjera $r$ sa središtem $S$

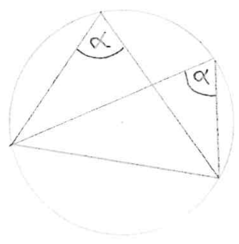
$k_S(T)$ = kružnica sa središtem $S$ koja prolazi točkom $T$.

**Teorem** (o obodnom i središnjem kutu): Središnji kut nad lukom kružnice jednak je dvostrukom obodnom kutu nad istim lukom.

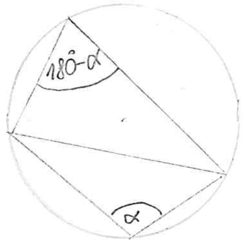



**Posljedice:**

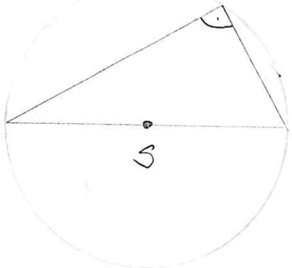
- obodni kutevi nad istim lukom su jednaki



- obodni kuteri nad suprotnim lukovima su suplementni (suma = 180°)




- Talesov teorem: obodni kut nad promjerom je pravi



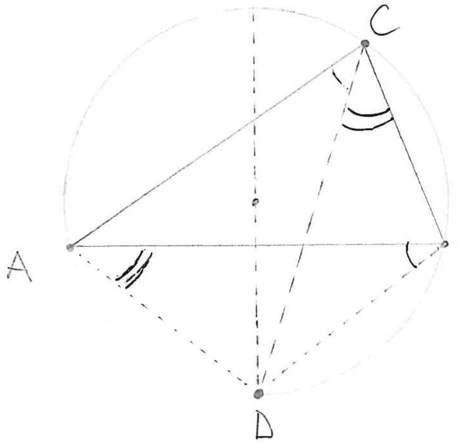

Vrijede i obrati tih tvrdnji:

- skup svih točaka koje zatvaraju pravi kut s dužinom $\overline{AB}$ je kružnica s promjerom $\overline{AB}$.
- skup svih točaka koje zatvaraju kut $\alpha$ ili $180°-\alpha$ s dužinom $\overline{AB}$ je kružnica s tetivom $\overline{AB}$

To su alternativne definicije kružnice.

zad 21.: Dokažite: simetrala kuta i simetrala nasuprotne stranice trokuta sijeku se u točki na opisanoj kružnici.

rj:



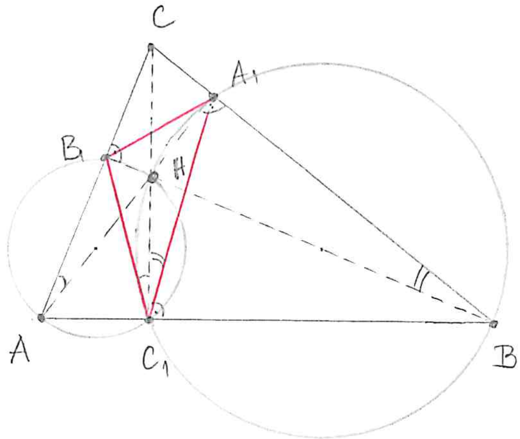

Neka simetrala stranice $\overline{AB}$ siječe opisanu kružnicu u točki D.

Treba dokazati $\measuredangle ACD = \measuredangle DCB$.

Trokut $\triangle ADB$ je jednakokračan, a to znači da su lukovi $\widehat{AD}$ i $\widehat{DB}$ sukladni $\Rightarrow$ obodni kuteri nad tim lukovima $\measuredangle ACD$ i $\measuredangle BCD$ su jednaki.

**zad 22.:** Dokažite da su visine trokuta $\Delta ABC$ ujedno simetrale kutova njegovog nožišnog trokuta.

**nj:** nožišni trokut = trokut čiji su vrhovi nožišta visina trokuta $\Delta ABC$.




Dokazat ćemo da je $\measuredangle B_1C_1C = \measuredangle CC_1A_1$.
(ostalo se dokazuje analogno).

Kako je $\measuredangle AB_1H = \measuredangle AC_1H = 90°$, to nam obrat Talesovog tm-a kaže da točke $A, C_1, H, B_1$ leže na kružnici s promjerom $\overline{AH}$. Obodni kutovi nad lukom $\widehat{HB_1}$ te kružnice su jednaki. Dakle,
$\measuredangle B_1C_1H = \measuredangle B_1AH = \measuredangle B_1AA_1 = 90° - \gamma$. Analogno se pokaže da točke $A_1, B, C_1, H$ leže na kružnici s promjerom $\overline{BH}$ pa su stoga kutovi nad lukom $\widehat{HA_1}$ $\measuredangle HC_1A_1$, $\measuredangle HBA_1$ jednaki.

Dakle, $\measuredangle A_1C_1H = \measuredangle HBA_1 = \measuredangle B_1BC = 90° - \gamma$.

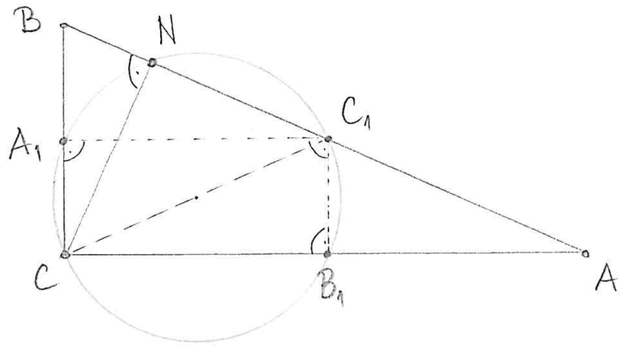
Stoga je $\measuredangle B_1C_1C = \measuredangle BC_1H = 90° - \gamma = \measuredangle HC_1A_1 = \measuredangle CC_1A_1$.

**Napomena:** Svakom trokutu se može opisati kružnica. No, ako je zadano četiri ili više točaka u ravnini, onda one ne moraju ležati na istoj kružnici.

**def:** Za točke koje leže na istoj kružnici kažemo da su **konciklične**, a za točke koje leže na istom pravcu kažemo da su **kolinearne**.

**Zad 23.:** Neka je $\Delta ABC$ pravokutni trokut s pravim kutem pri vrhu $C$. Dokažite da su polovišta stranica, vrh $C$ i nožište visine iz vrha $C$ konciklične točke.

**rj:**

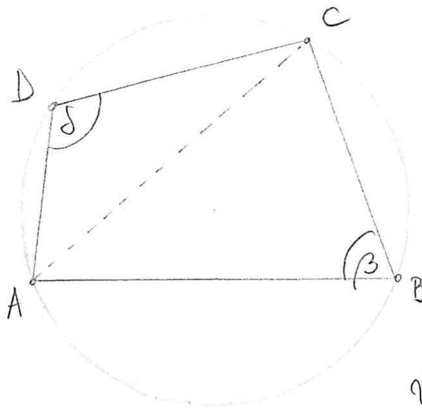



Neka su $A_1, B_1, C_1$ polovišta stranica. Kako su srednjice paralelne stranicama trokuta, to je $CA_1 \parallel B_1C_1$ & $CB_1 \parallel A_1C_1$.

Dakle $CB_1C_1A_1$ je paralelogram, a kako je $\measuredangle A_1CB_1 = 90°$, to je četverokut $CB_1C_1A_1$ pravokutnik. Po obratu Talesovog poučka pravokutniku se može opisati kružnica sa središtem na polovištu dijagonala.

$\measuredangle CNC_1 = 90°$, a promjer kružnice opisane pravokutniku $CB_1C_1A_1$ je $\overline{CC_1}$. Po obratu Talesovog poučka, točka $N$ također leži na kružnici s promjerom $\overline{CC_1}$.

Kada se četverokutu može opisati kružnica?




$\delta$ i $\beta$ su obodni kuteri nad suprotnim lukovima kružnice (određeni dijagonalom $\overline{AC}$) pa je $\beta + \delta = 180°$. Analogno se dokaže i $\alpha + \gamma = \beta + \delta = 180°$.

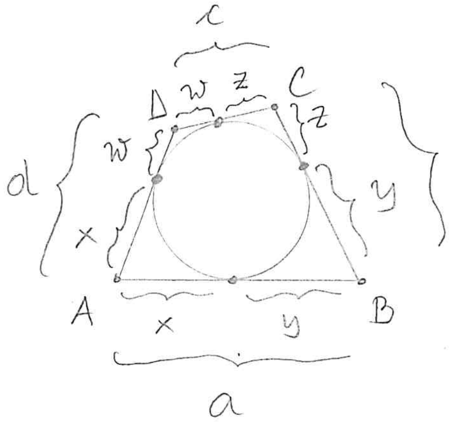
Vrijedi i obrat: ako je $\alpha + \gamma = \beta + \delta = 180°$, onda se četverokutu može opisati kružnica. Takvi se četverokuti zovu tetivni.

**Teorem:** Četverokut je tetivni ako i samo ako je zbroj nasuprotnih kutova jednak.

$\alpha + \beta + \gamma + \delta = 2 \cdot 180° = 360°$ pa iz $\alpha + \gamma = \beta + \delta \Rightarrow$ zbroj nasuprotnih kutova je $180°$.

Karakterizacija tetivnog četverokuta: četverokut je tetivni ako i samo ako se simetrale njegovih stranica sijeku u istoj točki.

Pod kojim uvjetom se četverokutu može upisati kružnica?




$$\left.\begin{aligned} a &= x+y \\ b &= y+z \\ c &= z+w \\ d &= w+x \end{aligned}\right\} \quad \begin{aligned} &a+c = x+y+z+w \\ &b+d = x+y+z+w \\ &\Rightarrow a+c = b+d \end{aligned}$$

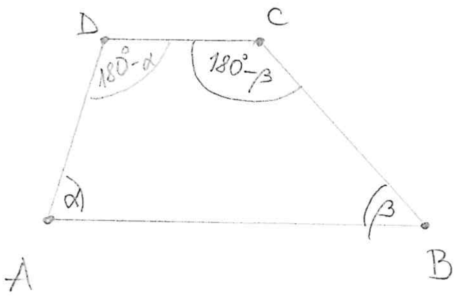
Vrijedi i obrat: ako je $a+c=b+d$, onda se četverokutu može upisati kružnica.

Takvi četverokuti zovu se tangencijalni.

Teorem: Četverokut je tangencijalni ako i samo ako je zbroj nasuprotnih stranica jednak.

Zad 24.: Dokažite da je visina trapeza kojemu se može opisati i upisati kružnica jednaka geometrijskoj sredini njegovih osnovica.

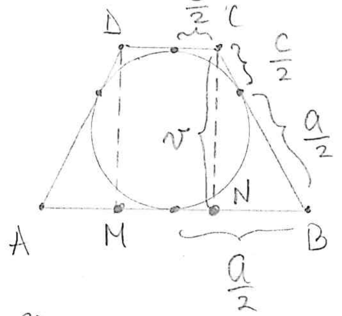
Rj:




$\measuredangle ADC = 180° - \measuredangle DAB = 180° - \alpha$

$\measuredangle DCB = 180° - \measuredangle ABC = 180° - \beta$

Trapez je tetivni $\Leftrightarrow$ $\alpha + 180° - \beta = 180°$ & $\beta + 180° - \alpha = 180°$ $\Leftrightarrow$ $\alpha = \beta$ $\Leftrightarrow$ je trapez jednakokračan.




$|AB| = a$

$|CD| = c$

$|BC| = |AD| = b$

$$b = \frac{a}{2} + \frac{c}{2} = \frac{a+c}{2}$$

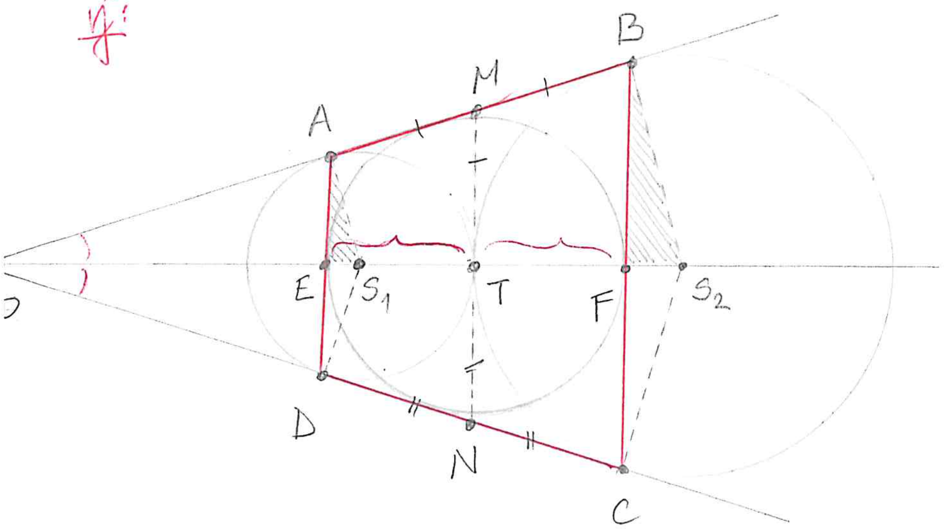
Neka je $N$ nožište visine iz vrha $C$, a $M$ nožište visine iz vrha $D$. Tada je $a = |AB| = |AM| + |MN| + |NB| = 2 \cdot |NB| + c \Rightarrow |NB| = \frac{1}{2}(a-c)$.

Sada je $v^2 = b^2 - \left(\frac{1}{2}(a-c)\right)^2 = \frac{1}{4}\left(\cancel{a^2} + 2ac + \cancel{c^2} - \cancel{a^2} + 2ac - \cancel{c^2}\right) = ac.$

$\Rightarrow v = \sqrt{ac}.$

**zad 25.:** Dvije kružnice polumjera $r$ i $R$ dodiruju se izvana. Njihove zajedničke tangente dodiruju ih u točkama $A, B$, odnosno $C, D$. Dokažite da se u četverokut $ABCD$ može upisati kružnica i odredite joj polumjer.

Rj:




Neka je $T$ točka dodira tih kružnica i neka tangenta na kružnice kroz točku $T$ siječe zajedničke tangente u točkama $M$ i $N$.

$ABCD$ je jednakokračan trapez. Doista, $|AB| = |PB| - |PA| =$
$= |PC| - |PD| = |DC|$ ($|PB| = |PC|$ jer je $\Delta PCS_2 \cong \Delta PBS_2$ po SSK$^>$ ($|BS_2| = |CS_2|$, $\sphericalangle PBS_2 = \sphericalangle PCS_2 = 90°$, $|PS_2| = |PS_2|$. Analogno se pokaže i $|PA| = |PD|$). Nadalje, $\Delta PAD \sim \Delta PBC$ (oba jednakokračna i $\sphericalangle APD = \sphericalangle BPC$) pa je $AD \parallel BC$.

Nadalje, $|AM| = |MT| = |MB| \Rightarrow M$ je polovište dužine $\overline{AB}$.
$|DN| = |NT| = |NC| \Rightarrow N$ je polovište dužine $\overline{CD}$.

$\Rightarrow \overline{MN}$ je srednjica trapeza $ABCD$ pa je $|MN| = \frac{|AD| + |BC|}{2}$.

S druge strane, $|MN| = |AB| = |CD|$ pa je $|AD| + |BC| =$
$= 2|MN| = |AB| + |CD| \Rightarrow ABCD$ je tangencijalni četverokut.

Uz oznake kao na slici, $EF$ je promjer njemu upisane kružnice.

$\Delta PAE \cong \Delta PED$ S-K-S $\left\{ \begin{array}{l} \sphericalangle APE = \sphericalangle DPE \\ |PA| = |PD| \\ PE \text{ zajednička} \end{array} \right.$ $\Rightarrow \sphericalangle PED = \sphericalangle PEA = 90°$

$|ES_1|=x$, $|FS_2|=y$

Tada je $|ET|=|TF| \Rightarrow r+x=R-y=\rho$

$\Delta ES_1A \sim \Delta FS_2B$ (KKK, paralelni kraci)

$\Rightarrow \frac{x}{y}=\frac{r}{R} \Rightarrow x=y\frac{r}{R}$ pa je $r+y\frac{r}{R}=R-y \Rightarrow y\left(1+\frac{r}{R}\right)=R-r$

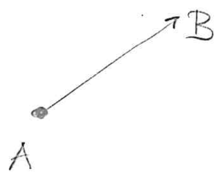
$\Rightarrow y=\frac{R}{R+r}(R-r)$.

Traženi radijus je $\rho=R-y=R\left(1-\frac{R-r}{R+r}\right)=R\,\frac{\cancel{R}+r-\cancel{R}+r}{R+r}=\frac{2rR}{r+R}$.

# 2. VEKTORI

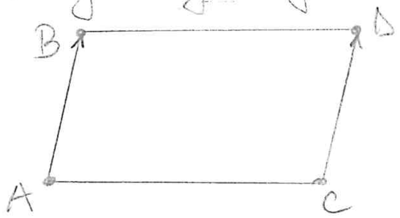
## 2.1. Uvod

orijentirana dužina = uređeni par dviju točaka




vektor = klasa ekvivalencije orijentiranih dužina

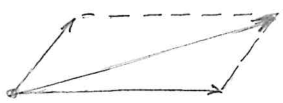
Relacija je definirana na sljedeći način:




$(A,B)\equiv(C,D)$ ako je polovište dužine $\overline{BC}$ = polovište dužine $\overline{AD}$.

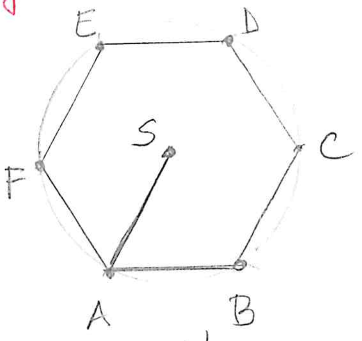
Vektore $\vec{AB}$ i $\vec{CD}$ smatramo jednakima. (paralelni su, imaju istu duljinu i istu orijentaciju)

- suprotni vektori imaju istu duljinu, paralelni su, ali imaju suprotnu orijentaciju.
- zbrajanje vektora



**Zad 1.:** Neka je ABCDEF pravilni šesterokut. Izrazite $\vec{AC}$ i $\vec{AF}$ pomoću $\vec{AB}$ i $\vec{AD}$.

rj:



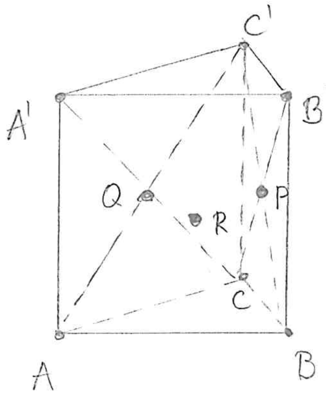

$$\vec{AC} = \vec{AS} + \vec{SC} = \frac{1}{2}\vec{AD} + \vec{AB}$$

$$\vec{AF} = \vec{AS} + \vec{SF} = \frac{1}{2}\vec{AD} - \vec{AB}$$

NE!

**zad 2.:** Neka je $ABCA'B'C'$ trostrana prizma, $P, Q, R$ redom središta stranica $BCC'B'$, $ACC'A'$, $ABB'A'$. Izrazite $\vec{B'C}$, $\vec{AQ}$ i $\vec{RP}$ preko $\vec{AB}$, $\vec{AC}$ i $\vec{AA'}$. Čine li $\{\vec{AB}, \vec{AC}, \vec{AA'}\}$ bazu od $\mathbb{R}^3$?

rj:




$$\vec{B'C} = \vec{B'B} + \vec{BC} = \vec{A'A} + \vec{BA} + \vec{AC} = -\vec{AB} + \vec{AC} - \vec{AA'},$$

$$\vec{AQ} = \frac{1}{2}\vec{AC'} = \frac{1}{2}(\vec{AC} + \vec{CC'}) = \frac{1}{2}\vec{AC} + \frac{1}{2}\vec{AA'},$$

$$\vec{RP} = \vec{RB} + \vec{BP} = \frac{1}{2}\vec{A'B} + \frac{1}{2}\vec{BC'} =$$

srednjica trokuta $A'BC'$

$$= \frac{1}{2}(\vec{A'A} + \vec{AB}) + \frac{1}{2}(\vec{BC} + \vec{CC'}) =$$

$$= \frac{1}{2}(\vec{AB} + \vec{BC}) + \frac{1}{2}\vec{A'A} + \frac{1}{2}\vec{AA'} = \frac{1}{2}\vec{AC}$$

$\vec{AB}$, $\vec{AC}$ i $\vec{AA'}$ su linearno nezavisni vektori u $\mathbb{R}^3$ (koji je trodim.) pa oni čine bazu.

Koji je zapis vektora $\vec{B'C}$, $\vec{AQ}$, $\vec{RP}$, $\vec{AB}$, $\vec{AC}$, $\vec{AA'}$ u toj bazi?

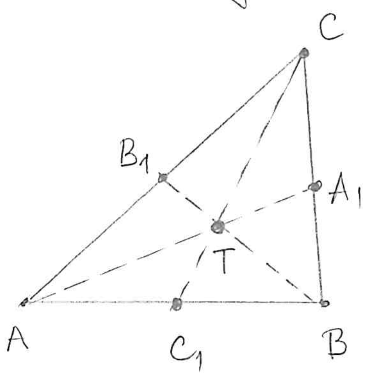
Zapis je sljedeći:

$\vec{B'C} = (-1, 1, -1)$, $\vec{AQ} = (0, \frac{1}{2}, \frac{1}{2})$, $\vec{RP} = (0, \frac{1}{2}, 0)$,

$\vec{AB} = (1, 0, 0)$, $\vec{AC} = (0, 1, 0)$, $\vec{AA'} = (0, 0, 1)$.

**zad3.:** Neka je $T$ težište trokuta $\Delta ABC$. Dokažite $\vec{AT} + \vec{BT} + \vec{CT} = \vec{0}$.

rj:




Neka su $A_1, B_1, C_1$ redom polovišta stranica $\overline{BC}$, $\overline{AC}$, $\overline{AB}$. Vrijedi $\frac{|AT|}{|TA_1|} = \frac{2}{1}$ pa je

$$\vec{AT} = \frac{2}{3}\vec{AA_1} = \frac{2}{3}(\vec{AB} + \vec{BA_1}) = \frac{2}{3}(\vec{AB} + \frac{1}{2}\vec{BC}) =$$

$$= \frac{2}{3}\vec{AB} + \frac{1}{3}(\vec{BA} + \vec{AC}) = \frac{2}{3}\vec{AB} - \frac{1}{3}\vec{AB} + \frac{1}{3}\vec{AC} =$$

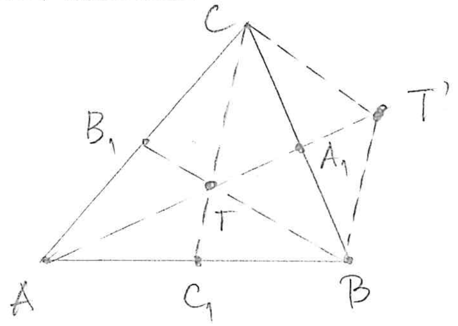
$$= \frac{1}{3}\vec{AB} + \frac{1}{3}\vec{AC}$$

$$\vec{BT} = \vec{BA} + \vec{AT} = -\vec{AB} + \frac{1}{3}\vec{AB} + \frac{1}{3}\vec{AC} = -\frac{2}{3}\vec{AB} + \frac{1}{3}\vec{AC}$$

$$\vec{CT} = \vec{CA} + \vec{AT} = -\vec{AC} + \frac{1}{3}\vec{AB} + \frac{1}{3}\vec{AC} = \frac{1}{3}\vec{AB} - \frac{2}{3}\vec{AC}$$

Sada je $\vec{AT} + \vec{BT} + \vec{CT} = \frac{1}{3}\vec{AB} + \frac{1}{3}\vec{AC} - \frac{2}{3}\vec{AB} + \frac{1}{3}\vec{AC} + \frac{1}{3}\vec{AB} - \frac{2}{3}\vec{AC} = \vec{0}$

Napomena: Kako se dokaže $\frac{|AT|}{|TA_1|} = \frac{2}{1}$ ?




$T'$ ... na pravcu $AA_1$ t.d. $|TA_1| = |T'A_1|$.

$|TA_1| = |A_1T'|$, $|BA_1| = |CA_1| \Rightarrow TBT'C$ je paralelogram $\Rightarrow BT' \parallel CT \Rightarrow BT' \parallel TC_1$

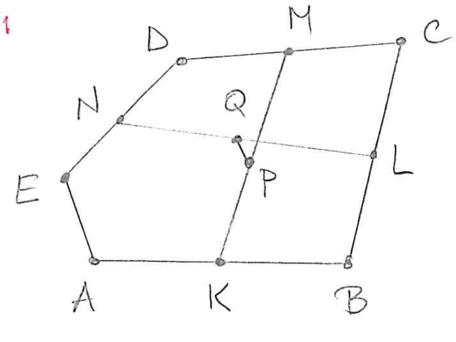
$\Delta ATC_1 \sim \Delta AT'B \Rightarrow \frac{|AT|}{|AT'|} = \frac{|AC_1|}{|AB|} = \frac{1}{2}$

$\Rightarrow \frac{|AT'|}{|AT|} = 2 \Rightarrow \frac{|AT| + 2|TA_1|}{|AT|} = 2 \Rightarrow \frac{|TA_1|}{|TA|} = \frac{1}{2}$

$\Rightarrow \frac{|AT|}{|TA_1|} = 2$.

Zad 4.: Neka je $ABCDE$ peterokut, $K, L, M, N$ polovišta stranica $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DE}$, $P, Q$ polovišta stranica $\overline{KM}$ i $\overline{LN}$. Pokažite da su $\overline{AE}$ i $\overline{PQ}$ paralelni i $|PQ| = \frac{1}{4}|AE|$.

4.




Dovoljno je dokazati $\vec{PQ} = \frac{1}{4}\vec{AE}$.

$$\vec{PQ} = \vec{PK} + \vec{KA} + \vec{AE} + \vec{EN} + \vec{NQ} =$$

$$= \frac{1}{2}\vec{MK} + \frac{1}{2}\vec{BA} + \vec{AE} + \frac{1}{2}\vec{ED} + \frac{1}{2}\vec{NL} =$$

$$= \frac{1}{2}(\vec{MC} + \vec{CB} + \vec{BK}) + \frac{1}{2}\vec{BA} + \vec{AE} + \frac{1}{2}\vec{ED}$$

$$+ \frac{1}{2}(\vec{NE} + \vec{EA} + \vec{AB} + \vec{BL}) =$$

$$= \frac{1}{4}\vec{DC} + \frac{1}{2}\vec{CB} + \frac{1}{4}\vec{BA} + \frac{1}{2}\vec{BA} + \vec{AE} + \frac{1}{2}\vec{ED} + \frac{1}{4}\vec{DE} + \frac{1}{2}\vec{EA} + \frac{1}{2}\vec{AB} + \frac{1}{4}\vec{BC} =$$

$$= \frac{1}{4}\vec{DC} + \frac{1}{4}\vec{CB} + \frac{1}{4}\vec{BA} + \frac{1}{2}\vec{AE} + \frac{1}{4}\vec{ED} = \frac{1}{4}(\vec{DC} + \vec{CB} + \vec{ED} + \vec{BA}) + \frac{1}{2}\vec{AE} =$$

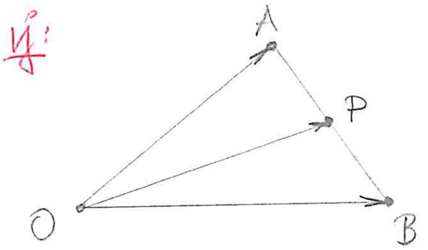
$$= \frac{1}{4}(\vec{DB} + \vec{ED} + \vec{BA}) + \frac{1}{2}\vec{AE} = \frac{1}{4}(\vec{DA} + \vec{ED}) + \frac{1}{2}\vec{AE} = \frac{1}{4}\vec{EA} + \frac{1}{2}\vec{AE} = \frac{1}{4}\vec{AE}.$$

## 2.1 Radijvektor

- odaberemo fiksnu točku ravnine O
- svaka točka A određuje vektor $\vec{a}=\overrightarrow{OA}$ kojeg zovemo radijvektor točke A.

**zad 5.:** Odredite radijvektor polovišta dužine $\overline{AB}$ ako znate radijvektore točaka A i B.

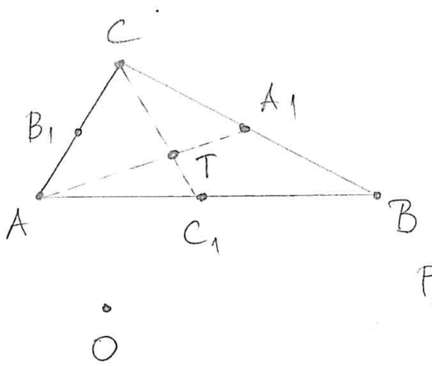
rj:




$$\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}=\overrightarrow{OA}+\frac{1}{2}\overrightarrow{AB}=\overrightarrow{OA}+\frac{1}{2}(\overrightarrow{AO}+\overrightarrow{OB})=$$
$$=\frac{1}{2}\overrightarrow{OA}+\frac{1}{2}\overrightarrow{OB}$$

**zad 6.:** Odredite radijvektor težišta trokuta ABC, ako su zadani radijvektori vrhova A, B i C.

rj:




$$\overrightarrow{OT}=\overrightarrow{OA}+\overrightarrow{AT}=\overrightarrow{OA}+\frac{2}{3}\overrightarrow{AA_1}=\overrightarrow{OA}+\frac{2}{3}(\overrightarrow{OA_1}+\overrightarrow{AO})$$
$$=\overrightarrow{OA}+\frac{2}{3}\left(\frac{1}{2}(\overrightarrow{OB}+\overrightarrow{OC})-\overrightarrow{OA}\right)=$$
$$=\frac{1}{3}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC})$$

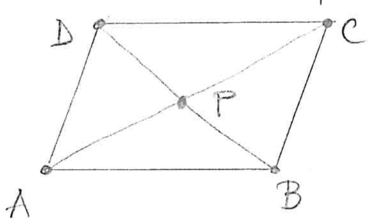
prethodni zadatak

**zad 7.:** Kažemo da P dijeli $\overline{AB}$ u omjeru $\lambda$ ako je $\overrightarrow{AP}=\lambda\overrightarrow{AB}$. Izračunajte radijvektor točke P ako su dani radijvektori A i B.

rj: $\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}=\overrightarrow{OA}+\lambda\overrightarrow{AB}=\overrightarrow{OA}+\lambda(\overrightarrow{AO}+\overrightarrow{OB})=(1-\lambda)\overrightarrow{OA}+\lambda\overrightarrow{OB}$

**zad 8.:** Dokažite da je ABCD paralelogram $\Longleftrightarrow$ mu se dijagonale raspolavljaju.

$\boxed{\Longrightarrow}$




P = polovište dužine $\overline{AC}$

Q = polovište dužine $\overline{BD}$

Neka je ABCD paralelogram. Treba pokazati P=Q.

$\vec{OP} = \frac{1}{2}(\vec{OA}+\vec{OC})$

$\vec{OQ} = \frac{1}{2}(\vec{OB}+\vec{OD}) = \frac{1}{2}(\vec{OA}+\vec{AB}) + \frac{1}{2}(\vec{OC}+\vec{CD}) = \frac{1}{2}(\vec{OA}+\vec{OC}) + \frac{1}{2}(\vec{AB}+\vec{CD})$

$= \vec{OP} + \frac{1}{2}(\vec{AB}+\vec{BA}) = \vec{OP} \quad \Rightarrow \quad P=Q.$

$\boxed{\Leftarrow}$ Pretpostavimo $P=Q$. Tada je $\vec{OP}=\vec{OQ}$.

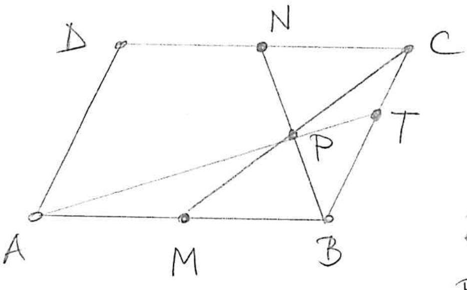
$\Rightarrow \frac{1}{2}(\vec{OA}+\vec{OC}) = \frac{1}{2}(\vec{OB}+\vec{OD}) \Rightarrow \vec{OA}+\vec{OC}=\vec{OB}+\vec{OD}$

$\Rightarrow \vec{OA}-\vec{OB} = \vec{OD}-\vec{OC} \Rightarrow \vec{BO}+\vec{OA} = \vec{CO}+\vec{OD} \Rightarrow \vec{BA}=\vec{CD}$

$\Rightarrow BA \parallel CD$ i $|BA|=|CD| \Rightarrow ABCD$ je paralelogram.

Zad 9.: Neka je $ABCD$ paralelogram, $M$ i $N$ polovišta stranica $\overline{AB}$ i $\overline{CD}$, $P$ sjecište pravaca $BN$ i $CM$, $T$ sjecište pravaca $AP$ i $BC$. U kojem omjeru točka $T$ dijeli $\overline{BC}$?

Rj:




Izrazimo sve preko $\vec{AB}$ i $\vec{AD}$.
Uočimo, $MBCN$ je paralelogram, a $P$ je sjecište njegovih dijagonala pa je $P$ polovište dužina $\overline{MC}$ i $\overline{BN}$.

$\vec{AP} = \vec{AM}+\vec{MP} = \frac{1}{2}\vec{AB} + \frac{1}{2}\vec{MC} = \frac{1}{2}\vec{AB} + \frac{1}{2}(\vec{MB}+\vec{BC}) = \frac{1}{2}\vec{AB} + \frac{1}{4}\vec{AB} + \frac{1}{2}\vec{AD}$

$= \frac{3}{4}\vec{AB} + \frac{1}{2}\vec{AD}.$

$T$ leži na sjecištu pravaca $AP$ i $BC$. Stoga $\exists \lambda, \mu \in \mathbb{R}$ t.d.

$\vec{AT} = \mu\vec{AP}$ i $\vec{BT} = \lambda\vec{BC}$.

$\mu\vec{AP} = \vec{AT} = \vec{AB}+\vec{BT} = \vec{AB} + \lambda\vec{BC}$

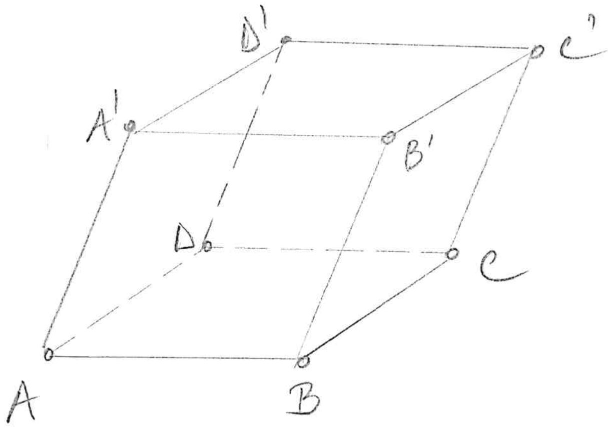
$\mu\left(\frac{3}{4}\vec{AB}+\frac{1}{2}\vec{AD}\right) = \vec{AB}+\lambda\vec{AD} \Rightarrow \frac{3}{4}\mu\vec{AB} + \frac{1}{2}\mu\vec{AD} = \vec{AB}+\lambda\vec{AD}$

Kako je $(\vec{AB}, \vec{AD})$ baza od $\mathbb{R}^2$, to je $\frac{3}{4}\mu = 1$ i $\frac{1}{2}\mu = \lambda$

$\Rightarrow \mu = \frac{4}{3}, \lambda = \frac{2}{3}$ Dakle, $\vec{BT} = \frac{2}{3}\vec{BC} \Rightarrow \frac{|BT|}{|TC|} = \frac{2}{1}$

**Zad 10.:** Neka je $ABCDA'B'C'D'$ paralelepiped. Dokažite da $\overline{AC'}$ siječe $\Delta BDA'$ u težištu.

**Rj:**




Neka je $T$ težište $\Delta BDA'$.

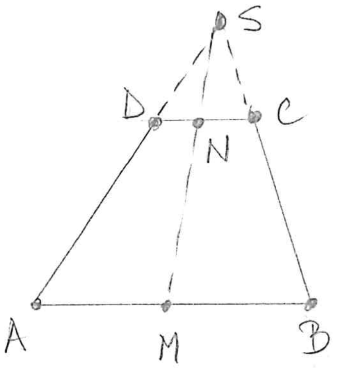
Prema zad 6. vrijedi

$$\overrightarrow{AT} = \frac{1}{3}\left(\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{AA'}\right)$$

$$\overrightarrow{AC'} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CC'} = \overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{AA'} = 3\cdot\overrightarrow{AT} \Rightarrow T \text{ leži na } \overline{AC'}.$$

**Zad 11.:** Neka je $ABCD$ trapez. Polovišta osnovica i sjecište produžetka krakova leže na istom pravcu. Dokažite!

**Rj:**




$M =$ polovište od $\overline{AB}$

$N =$ polovište od $\overline{CD}$

$S = AD \cap BC$

$\overrightarrow{DC} = \alpha\cdot\overrightarrow{AB}$

Izrazimo sve preko $\overrightarrow{AB}$ i $\overrightarrow{AD}$.

$$\overrightarrow{MN} = \overrightarrow{MA}+\overrightarrow{AD}+\overrightarrow{DN} = -\frac{1}{2}\overrightarrow{AB} + \overrightarrow{AD} + \frac{1}{2}\alpha\overrightarrow{AB} = \frac{-1+\alpha}{2}\overrightarrow{AB}+\overrightarrow{AD}$$

$$\left.\begin{array}{l}\overrightarrow{MS} = \overrightarrow{MA}+\overrightarrow{AS} = \overrightarrow{MA}+\lambda\overrightarrow{AD}\\ \overrightarrow{MS} = \overrightarrow{MB}+\overrightarrow{BS} = \overrightarrow{MB}+\mu\cdot\overrightarrow{BC}\end{array}\right\} \Rightarrow \overrightarrow{MA}+\lambda\overrightarrow{AD} = \overrightarrow{MB}+\mu\cdot\overrightarrow{BC}$$

$$\Rightarrow -\frac{1}{2}\overrightarrow{AB}+\lambda\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AB}+\mu\left(\overrightarrow{BA}+\overrightarrow{AD}+\overrightarrow{DC}\right)$$

$$= \left(\frac{1}{2}-\mu\right)\overrightarrow{AB}+\mu\overrightarrow{AD}+\mu\alpha\overrightarrow{AB}$$

$$= \left(\frac{1}{2}+\mu(-1+\alpha)\right)\overrightarrow{AB}+\mu\overrightarrow{AD}$$

$$\Rightarrow -\frac{1}{2} = \frac{1}{2}+\mu(-1+\alpha), \quad \lambda=\mu$$

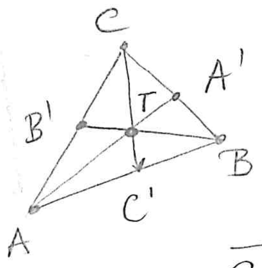
$$\Rightarrow \mu = \frac{1}{1-\alpha} = \lambda \Rightarrow \overrightarrow{MS} = -\frac{1}{2}\overrightarrow{AB}+\frac{1}{1-\alpha}\overrightarrow{AD} = \frac{1}{1-\alpha}\left(\frac{-1+\alpha}{2}\overrightarrow{AB}+\overrightarrow{AD}\right) =$$

$$= \frac{1}{1-\alpha}\overrightarrow{MN} \Rightarrow M, N \text{ i } S \text{ su kolinearne}$$

DZ: Probajte sve izraziti preko $\overrightarrow{SD}$ i $\overrightarrow{SC}$.

**Zad 12.:** Dokažite da se težišnice trokuta sijeku u istoj točki.

**Rj:**




$$T = \overline{BB'} \cap \overline{AA'}$$

$$\overrightarrow{CT} = \overrightarrow{CA} + \overrightarrow{AT} = \overrightarrow{CA} + \lambda \cdot \overrightarrow{AA'}$$

$$\overrightarrow{CT} = \overrightarrow{CB} + \overrightarrow{BT} = \overrightarrow{CB} + \mu \cdot \overrightarrow{BB'}$$

$$\overrightarrow{CA} + \lambda \cdot \overrightarrow{AA'} = \overrightarrow{CB} + \mu \overrightarrow{BB'}$$

$$\Rightarrow \overrightarrow{CA} + \lambda\left(\overrightarrow{AC} + \overrightarrow{CA'}\right) = \overrightarrow{CB} + \mu\left(\overrightarrow{CB'} - \overrightarrow{CB}\right)$$

$$\Rightarrow \overrightarrow{CA} + \lambda\left(\tfrac{1}{2}\overrightarrow{CB} - \overrightarrow{CA}\right) = \overrightarrow{CB} + \mu\left(\tfrac{1}{2}\overrightarrow{CA} - \overrightarrow{CB}\right)$$

$$\Rightarrow (1-\lambda)\overrightarrow{CA} + \tfrac{1}{2}\lambda\overrightarrow{CB} = \tfrac{1}{2}\mu\overrightarrow{CA} + (1-\mu)\overrightarrow{CB}$$

$$\Rightarrow 1-\lambda = \tfrac{1}{2}\mu \Rightarrow \mu = 2-2\lambda$$

$$\tfrac{1}{2}\lambda = 1-\mu \Rightarrow \lambda = 2-2\mu = 2-4+4\lambda = 4\lambda - 2 \Rightarrow \boxed{\lambda = \frac{2}{3}}$$

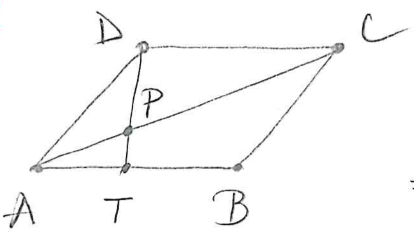
$$\overrightarrow{CT} = \frac{1}{3}\overrightarrow{CA} + \frac{1}{3}\overrightarrow{CB}$$

$$\overrightarrow{CC'} = \overrightarrow{CA} + \overrightarrow{AC'} = \overrightarrow{CA} + \tfrac{1}{2}\overrightarrow{AB} = \overrightarrow{CA} + \tfrac{1}{2}\left(\overrightarrow{AC} + \overrightarrow{CB}\right) = \tfrac{1}{2}\overrightarrow{CA} + \tfrac{1}{2}\overrightarrow{CB}$$

$\Rightarrow \overrightarrow{CT} = \frac{2}{3}\overrightarrow{CC'} \Rightarrow C, T$ i $C'$ su kolinearne.

**Zad 13.:** Neka je $ABCD$ paralelogram, $T \in \overline{AB}$, $\overrightarrow{AT} = \frac{1}{n}\overrightarrow{AB}$, $P = \overline{AC} \cap \overline{TD}$. U kojem omjeru $P$ dijeli $\overline{AC}$?

**Rj:**




$$\overrightarrow{AP} = \lambda \overrightarrow{AC}$$

$$\overrightarrow{AP} = \overrightarrow{AD} + \overrightarrow{DP} = \overrightarrow{AD} + \mu \cdot \overrightarrow{DT}$$

$$\Rightarrow \lambda\overrightarrow{AC} = \overrightarrow{AD} + \mu \cdot \overrightarrow{DT}$$

$$\Rightarrow \lambda\left(\overrightarrow{AB} + \overrightarrow{BC}\right) = \overrightarrow{AD} + \mu\left(\overrightarrow{DA} + \tfrac{1}{n}\overrightarrow{AB}\right)$$

$$\Rightarrow \lambda\overrightarrow{AB} + \lambda\overrightarrow{AD} = \frac{\mu}{n}\overrightarrow{AB} + (1-\mu)\overrightarrow{AD} \Rightarrow \begin{array}{l} \lambda = \frac{1}{n}\cdot\mu \\ \lambda = 1-\mu \end{array}$$

$$\Rightarrow \frac{\mu}{n} = 1-\mu \Rightarrow \mu = \frac{n}{n+1} \Rightarrow \lambda = \frac{1}{n+1} \Rightarrow \frac{|AP|}{|PC|} = \frac{1}{n}.$$

## 2. 2. Koordinatizacija

def: Vektori $\vec{a}$ i $\vec{b}$ su kolinearni ako su istog smjera, tj. ako postoji kolinearne točke $O, A$ i $B$ t.d. $\vec{a} = \overrightarrow{OA}$, $b = \overrightarrow{OB}$.

Vektori $\vec{a}, \vec{b}$ i $\vec{c}$ su komplanarni ako leže u istoj ravnini, tj. ako postoje točke $O, A, B, C$ koje su komplanarne točke t.d. $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$, $\vec{c} = \overrightarrow{OC}$.

Zad 14.: Neka su $\vec{i}, \vec{j}, \vec{k}$ linearno nezavisni vektori.

a) Jesu li vektori $\vec{a} = 2\vec{i} + 3\vec{j} + \vec{k}$ i $\vec{b} = -\vec{i} - \frac{3}{2}\vec{j} - \frac{1}{2}\vec{k}$ kolinearni?

b) Jesu li $\vec{a}$ i $\vec{b}$ komplanarni?

c) Jesu li $\vec{f} = \vec{i} + \vec{j}$, $\vec{g} = \vec{i} - \vec{j}$, $h = 2\vec{j}$ komplanarni?

d) Može li se $(3, \frac{1}{2}, \frac{3}{2})$ prikazati kao linearna kombinacija vektora $(1,1,1)$ i $(-2,3,1)$?

Rj: a) Da, $\vec{a} = -2\vec{b}$

b) Da, bilo koja 2 vektora su uvijek komplanarna (jer za svake tri točke uvijek postoji ravnina koja ih sadrži)

c) Da, jer su sva tri linearna kombinacija vektora $\vec{i}$ i $\vec{j}$ pa leže u ravnini određenoj s ta tri vektora.

d) $(3, \frac{1}{2}, \frac{3}{2}) = \alpha(1,1,1) + \beta(-2,3,1) = (\alpha - 2\beta, \alpha + 3\beta, \alpha + \beta)$

$$\Rightarrow \left.\begin{array}{l} \left.\begin{array}{l} 3 = \alpha - 2\beta \\ \frac{1}{2} = \alpha + 3\beta \end{array}\right\} \frac{7}{2} = 2\alpha + \beta \\ \frac{3}{2} = \alpha + \beta \end{array}\right\} \boxed{\alpha = 2} \Rightarrow \boxed{\beta = -\frac{1}{2}} \quad \text{DA?}$$

**Zad 15.:** Jesu li vektori:

a) $(2,-3),(-4,6)$; b) $(2,0),(0,1)$; c) $(2,-3,1),(4,6,-2)$;

d) $(1,1,1),(1,2,3),(0,1,2)$; e) $(1,1,1),(1,0,0),(0,1,0)$ linearno zavisni ili linearno nezavisni?

Rj: e) Dani vektori su linearno nezavisni ako i samo ako matrica koja ih sadrži kao retke ima puni rang, odnosno ako i samo ako je regularna, tj. ako joj je determinanta jednaka 0.

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \Longrightarrow \text{matrica je regularna}$$

$\Longrightarrow$ vektori su linearno nezavisni

**Zad 16.:** U ovisnosti o parametrima $m, n \in \mathbb{R}$ ispitajte linearnu nezavisnost sljedećih vektora:

a) $(1,m,1),(m,1,-1),(1,-1,1)$

b) $(m,m,2),(1,1,1),(2,2,1)$

c) $(1,m,1),(n,1,1),(1,1,1)$

Rj: b) $\begin{vmatrix} m & m & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0 \Rightarrow$ vektori su linearno zavisni.

a) $$\left|\begin{matrix} 1 & m & 1 \\ m & 1 & -1 \\ 1 & -1 & 1 \end{matrix}\right|\begin{matrix} 1 & m \\ m & 1 \\ 1 & -1 \end{matrix} = \cancel{1} - m - m - m^2 \cancel{-1} - 1 = -(m^2+2m+1) =$$
$$= -(m+1)^2$$

vektori su lin. nez. $\Longleftrightarrow \det \neq 0 \Longleftrightarrow m \neq -1$.

c) $$\left|\begin{matrix} 1 & m & 1 \\ n & 1 & 1 \\ 1 & 1 & 1 \end{matrix}\right|\begin{matrix} 1 & m \\ n & 1 \\ 1 & 1 \end{matrix} = \cancel{1} + m + n - mn \cancel{-1} - 1 = m - 1 + n(1-m) = (m-1)(1-n)$$

lin. nez. za $m \neq 1$ i $n \neq 1$

lin. zavisni za $m = 1$ ili $n = 1$.

# 2.4. Skalarni produkt

$$\vec a\cdot\vec b=\underbrace{|\vec a|\cdot|\vec b|\cdot\cos\measuredangle(\vec a,\vec b)}_{\in\mathbb{R}}\qquad \measuredangle(\vec a,\vec b)=[0,\pi]$$

Svojstva:

| | |
|---|---|
| $\vec a^{\,2}=|\vec a|^2\geq 0$ | nenegativnost |
| $\vec a^{\,2}=0 \iff \vec a=\vec 0$ | definitnost |
| $\vec a\cdot\vec b=\vec b\cdot\vec a$ | komutativnost |
| $(\lambda\vec a)\cdot\vec b=\lambda(\vec a\cdot\vec b)$ | kvaziasocijativnost |
| $\vec a(\vec b+\vec c)=\vec a\vec b+\vec a\vec c$ | distributivnost prema zbrajanju. |

**Zad 17.:** Neka je $(\vec i,\vec j,\vec k)$ ONB u $\mathbb{R}^3$ (tj. $|\vec i|=|\vec j|=|\vec k|=1$, $\vec i\perp\vec j\perp\vec k\perp\vec i$). Odredite skalarni produkt vektora $\vec i-\vec j+\vec k$ i $2\vec i-3\vec j-\vec k$.

NE!

Rj: $(\vec i-\vec j+\vec k)(2\vec i-3\vec j-\vec k)=2\vec i\cdot\vec i-3\vec i\cdot\vec j-\vec i\cdot\vec k-2\vec j\cdot\vec i+3\vec j\cdot\vec j+\vec j\cdot\vec k$

distributivnost

$+2\vec k\cdot\vec i-3\vec k\cdot\vec j-\vec k\cdot\vec k$

$=2+3-1=4$

Ako su vektori zapisani u ONB, onda vrijedi

$$(x_1,x_2,x_3)\cdot(y_1,y_2,y_3)=x_1y_1+x_2y_2+x_3y_3$$

Ako nisu zapisani u ONB, onda to ne mora vrijediti.

**Zad 18.:** Zadani su vektori $\vec p$ i $\vec q$ t.d. $|\vec p|=2$, $|\vec q|=\sqrt3$, $\measuredangle(\vec p,\vec q)=\frac{\pi}{6}$. Odredite $|\vec p-2\vec q|$.

DA!

Rj: $|\vec p-2\vec q|^2=(\vec p-2\vec q)\cdot(\vec p-2\vec q)=\vec p^{\,2}-2\vec p\cdot\vec q-2\overbrace{\vec q\cdot\vec p}^{=\vec p\cdot\vec q}+4\vec q^{\,2}=$

$=|\vec p|^2-4\vec p\cdot\vec q+4|\vec q|^2=4-4\cdot2\cdot\sqrt3\cdot\cos\frac{\pi}{6}+4\cdot3=$

$=4-4\cdot\cancel{2}\cdot\sqrt3\cdot\frac{\sqrt3}{\cancel{2}}+12=4-12+12=4$

**Zad (DZ):** Dane su točke $A$, $B$ i $C$ u prostoru s koordinatama $A=(1,2,3)$, $B=(3,2,1)$, $C=(1,4,1)$. Pokažite da je $\Delta ABC$ jednakostraničan tako da pokažete:

NE!

(2.) U trokutu OAB je zadano $\overrightarrow{OA}=(1,2,1)$, $\overrightarrow{OB}=(-6,3,-3)$. Izračunajte duljinu težišnice i simetrale iz vrha O.

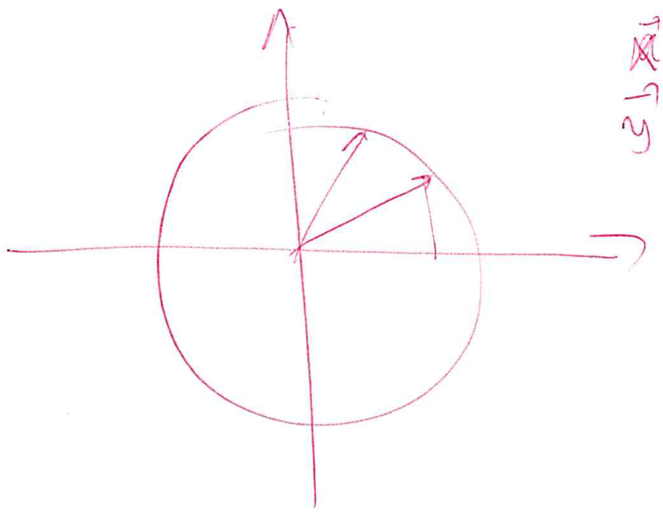
(3.) Dan je trokut $\triangle ABC$ s koordinatama $A=(1,1,1)$, $B=(2,4,3)$ i $C=(1,0,4)$. Odredite duljinu visine na stranicu $\overline{AB}$.

(4.) Koristeći vektore dokažite formulu

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

(5.) b) ~~Izvedite formulu za~~ Odredite ortogonalnu projekciju vektora $\vec{x}=(8,-4,3)$ na ravninu razapetu vektorima $\vec{a}=(1,2,1)$, $\vec{b}=(1,1,0)$

a) Izvedite formulu za ortogonalnu projekciju vektora $\vec{x}$ na vektor $\vec{a}$.



$\vec{x}=\cos\alpha\,\vec{i}+\sin\alpha\,\vec{j}$

$\vec{y}=\cos\beta\,\vec{i}+\sin\beta\,\vec{j}$

$\vec{x}\cdot\vec{y}=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$\vec{x}\cdot\vec{y}=1\cdot1\cdot\cos(\alpha-\beta)$

MATIČNI BROJ STUDENTA | IME I PREZIME | BROJ BODOVA

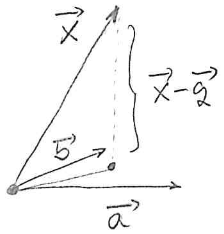
# ELEMENTARNA MATEMATIKA 2

Popravni kolokvij – 21. lipnja 2012.

## ZADATAK 4

Na stranici $\overline{AC}$ trokuta $ABC$ leži točka $M$ za koju je $|AM| : |MC| = 2 : 1$, a na stranici $\overline{BC}$ točka $N$ za koju je $|BN| : |NC| = 3 : 1$. Pravci $AN$ i $BM$ sijeku se u točki $T$. U kojem omjeru točka $T$ dijeli dužinu $\overline{AN}$?

b)




$\vec{q}=\lambda\vec{a}+\mu\vec{b}$

$\vec{x}-\vec{q}\perp\vec{a}$ & $\vec{x}-\vec{q}\perp\vec{b}$

$\vec{q}=(\lambda+\mu,2\lambda+\mu,\lambda)$

$\vec{x}-\vec{q}=(8-\lambda-\mu,-4-2\lambda-\mu,3-\lambda) \Rightarrow 0=(\vec{x}-\vec{q})\cdot\vec{a}=8-\lambda-\mu$

$+2(-4-2\lambda-\mu)+1\cdot(3-\lambda)=\cancel{8}-\lambda-\mu-\cancel{8}-4\lambda-2\mu+3-\lambda=-6\lambda-3\mu+3$

$\Rightarrow 6\lambda+3\mu=3.$

$0=(\vec{x}-\vec{q})\cdot\vec{b}=8-\lambda-\mu-4-2\lambda-\mu=4-3\lambda-2\mu \Rightarrow 3\lambda+2\mu=4$

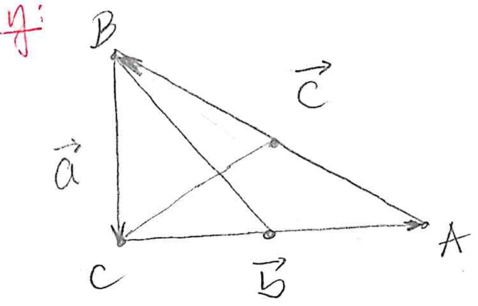
$\Rightarrow 6\lambda+4\mu=8 \Rightarrow \boxed{\mu=5} \Rightarrow \boxed{\lambda=-2}$

$\vec{q}=(-2,-4,-2)+(5,5,0)=(3,1,-2)$

**Zad 23.:** U pravokutnom trokutu ABC duljine stranica trokuta se odnose kao $1:\sqrt{2}:\sqrt{3}$. Dokažite da su dvije njegove težišnice okomite.

DA!

Rj:




$|\vec{a}|=k$

$|\vec{b}|=\sqrt{2}k$

$|\vec{c}|=\sqrt{3}k$

$\vec{t_a}=-\frac{1}{2}\vec{a}-\vec{b}$

$\vec{t_b}=\vec{a}+\frac{1}{2}\vec{b}$

$\vec{t_c}=\vec{b}+\frac{1}{2}\vec{c}=\vec{b}+\frac{1}{2}(-\vec{a}-\vec{b})=$

$=-\frac{1}{2}\vec{a}+\frac{1}{2}\vec{b}$

$\vec{t_b}\cdot\vec{t_c}=(\vec{a}+\frac{1}{2}\vec{b})\cdot(-\frac{1}{2}\vec{a}+\frac{1}{2}\vec{b})=-\frac{1}{2}|\vec{a}|^2+\frac{1}{4}\vec{a}\cdot\vec{b}+\frac{1}{4}|\vec{b}|^2=$

$=-\frac{1}{2}k^2+\frac{1}{4}\cdot 2k^2=0 \Rightarrow \vec{t_b}\cdot\vec{t_c}=\vec{0}.$

# 2.5. Vektorski produkt

**def:** Vektorski produkt vektora $\vec{a}$ i $\vec{b}$ je vektor $\vec{a}\times\vec{b}$ takav da vrijedi:

- ako su $\vec{a}$ i $\vec{b}$ kolinearni, onda je $\vec{a}\times\vec{b}=\vec{0}$.
- ako $\vec{a}$ i $\vec{b}$ nisu kolinearni, onda je
  - $\vec{a}\times\vec{b}$ okomit na $\vec{a}$ i na $\vec{b}$
  - modul $|\vec{a}\times\vec{b}|=|\vec{a}|\cdot|\vec{b}|\cdot\sin\angle(\vec{a},\vec{b})$
  - orijentacija od $\vec{a}\times\vec{b}$ je takva da $(\vec{a},\vec{b},\vec{a}\times\vec{b})$ čine DESNU BAZU.

**Svojstva:**

1.) $(\lambda\vec{a})\times\vec{b}=\lambda(\vec{a}\times\vec{b})$ kvaziasocijativnost

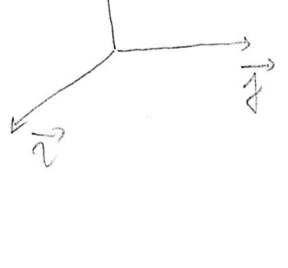
2.) $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ distributivnost prema zbrajanju

3.) $\vec{b}\times\vec{a}=-(\vec{a}\times\vec{b})$ antikomutativnost

4.) $\vec{a}\times\vec{a}=\vec{0}$

Asocijativnost ne vrijedi. Uzmimo vektore $\vec{a},\vec{b}$ i $\vec{c}$ koji su linearno nezavisni t.d. $\vec{c}\perp\vec{a}$, $\vec{c}\perp\vec{b}$ i $\vec{a}\not\perp\vec{b}$. Tada je vektor $\vec{a}\times\vec{b}$ okomit na $\vec{a}$ i na $\vec{b}$, a kako je $\vec{c}\perp\vec{a}$ i $\vec{c}\perp\vec{b}$, onda su vektori $\vec{a}\times\vec{b}$ i $\vec{c}$ kolinearni pa je $(\vec{a}\times\vec{b})\times\vec{c}=\vec{0}$.

S druge strane $\vec{a}$ i $\vec{b}\times\vec{c}$ nisu kolinearni jer je u protivnom $\vec{a}$ okomit na $\vec{b}$ i na $\vec{c}$, a po pretpostavci $\vec{a}\not\perp\vec{b}$. Stoga je $\sin(\vec{a},(\vec{b}\times\vec{c}))\neq 0 \Rightarrow |\vec{a}\times(\vec{b}\times\vec{c})|\neq 0 \Rightarrow \vec{a}\times(\vec{b}\times\vec{c})\neq\vec{0}$.




$\vec{i}\times\vec{j}=\vec{k}$

$\vec{j}\times\vec{k}=\vec{i}$

$\vec{k}\times\vec{i}=\vec{j}$

Za $\vec{a}=(a_1,a_2,a_3)$ i $\vec{b}=(b_1,b_2,b_3)$ (u ONB $\vec{i},\vec{j},\vec{k}$) vrijedi

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

**Zad 24.:** Izračunajte duljine vektora:

DA! a) $\vec{i}\times(\vec{j}+\vec{k})-\vec{j}\times(\vec{i}+\vec{k})+\vec{k}\times(\vec{i}+\vec{j}+\vec{k})$

b) $(2\vec{a}+\vec{b})\times(\vec{a}+2\vec{b})$, ako je $|\vec{a}|=1$, $|\vec{b}|=2$, $\sphericalangle(\vec{a},\vec{b})=60^\circ$

**Rj:** a) $\vec{i}\times(\vec{j}+\vec{k})-\vec{j}\times(\vec{i}+\vec{k})+\vec{k}\times(\vec{i}+\vec{j}+\vec{k})=$

$=\vec{i}\times\vec{j}+\vec{i}\times\vec{k}-\vec{j}\times\vec{i}-\vec{j}\times\vec{k}+\vec{k}\times\vec{i}+\vec{k}\times\vec{j}+\vec{k}\times\vec{k}=$

$=\vec{k}-\cancel{\vec{j}}+\vec{k}-\vec{i}+\cancel{\vec{j}}-\vec{i}=2(\vec{k}-\vec{i})=2(0,0,1)-2(1,0,0)=$

$=(-2,0,2)$

Duljina danog vektora je $\sqrt{(-2)^2+2^2}=2\sqrt{2}$

b) $(2\vec{a}+\vec{b})\times(\vec{a}+2\vec{b})=2\vec{a}\times\vec{a}+4\vec{a}\times\vec{b}+\vec{b}\times\vec{a}+2\vec{b}\times\vec{b}=3\,\vec{a}\times\vec{b}$

$|3\vec{a}\times\vec{b}|=3\,|\vec{a}\times\vec{b}|=3|\vec{a}|\cdot|\vec{b}|\cdot\sin\sphericalangle(\vec{a},\vec{b})=6\cdot\frac{\sqrt{3}}{2}=3\sqrt{3}$

Modul vektorskog produkta $|\vec{a}\times\vec{b}|=|\vec{a}|\cdot|\vec{b}|\sin\sphericalangle(\vec{a},\vec{b})$ jednak je površini paralelograma razapetog tim vektorima.

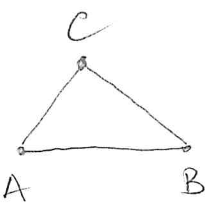

**Zad (DZ):** Izračunajte površinu paralelograma razapetog vrhovima

DA! $A=(1,1,0)$, $B=(4,1,0)$, $C=(2,2,0)$, $D=(5,2,0)$.

**Zad 25.:** Odredite omjer površine $\Delta ABC$ i trokuta razapetog

DA! težišnicama $\Delta ABC$.

**Rj:**




$\vec{AB}=\vec{c}$, $\vec{BC}=\vec{a}$, $\vec{CA}=\vec{b}$

$\vec{t_a}=\vec{AC}+\frac{1}{2}\vec{CB}=-\vec{b}-\frac{1}{2}\vec{a}$

$\vec{t_b}=\vec{BC}+\frac{1}{2}\vec{CA}=\vec{a}+\frac{1}{2}\vec{b}$

$\vec{t_c}=\frac{1}{2}(\vec{CA}+\vec{CB})=\frac{1}{2}(\vec{b}-\vec{a})=-\frac{1}{2}\vec{a}+\frac{1}{2}\vec{b}$

Vektori težišnica zaista čine trokut jer je $\vec{t_a}+\vec{t_b}+\vec{t_c}=\vec{0}$

$P_{ABC} = \frac{1}{2}|\vec{a}\times\vec{b}|$ (pola paralelograma)

$$P_{\text{težišnice}} = \frac{1}{2}|\vec{t_a}\times\vec{t_b}| = \frac{1}{2}\left|\left(-\frac{1}{2}\vec{a}-\vec{b}\right)\times\left(\vec{a}+\frac{1}{2}\vec{b}\right)\right| =$$

$$= \frac{1}{2}\left|-\frac{1}{2}\vec{a}\times\vec{a}-\frac{1}{4}\vec{a}\times\vec{b}-\vec{b}\times\vec{a}-\frac{1}{2}\vec{b}\times\vec{b}\right| = \frac{1}{2}\cdot\frac{3}{4}\cdot|\vec{a}\times\vec{b}| =$$

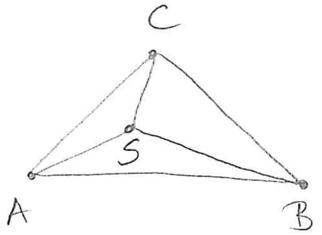
$$= \frac{3}{4}P_{ABC}$$

**Zad 26.:** Točka S je unutar $\triangle ABC$. Neka su $P_1$, $P_2$ i $P_3$ površine trokuta $\triangle SBC$, $\triangle SCA$, $\triangle SAB$. Dokažite:

DA!

$P_1\vec{SA}+P_2\vec{SB}+P_3\vec{SC}=\vec{0}$.

Rj:




$P_1 = \frac{1}{2}|\vec{SB}\times\vec{SC}|$, $P_2 = \frac{1}{2}|\vec{SC}\times\vec{SA}|$, $P_3 = \frac{1}{2}|\vec{SA}\times\vec{SB}|$

Prikažimo $\vec{SC}$ kao linearnu kombinaciju vektora $\vec{SA}$ i $\vec{SB}$.

$$\vec{SC} = \alpha\vec{SA}+\beta\vec{SB} \quad / \times \vec{SB}$$

$$\underbrace{\vec{SC}\times\vec{SB}}_{\text{u papir}} = \alpha\underbrace{(\vec{SA}\times\vec{SB})}_{\text{iz papira}}$$

(istog su smjera, suprotne orijentacije)

$$\Rightarrow \alpha = -\frac{|\vec{SC}\times\vec{SB}|}{|\vec{SA}\times\vec{SB}|} = -\frac{2P_1}{2P_3} = -\frac{P_1}{P_3}$$

$$\vec{SC} = \alpha\vec{SA}+\beta\vec{SB} \ / \times\vec{SA} \Rightarrow \underbrace{\vec{SC}\times\vec{SA}}_{\text{iz papira}} = \beta\underbrace{\vec{SB}\times\vec{SA}}_{\text{u papir}} \Rightarrow \beta = -\frac{|\vec{SC}\times\vec{SA}|}{|\vec{SB}\times\vec{SA}|}$$

$$\Rightarrow \beta = -\frac{2P_2}{2P_3} = -\frac{P_2}{P_3}$$

$$\vec{SC} = -\frac{P_1}{P_3}\vec{SA}-\frac{P_2}{P_3}\vec{SB} \Rightarrow P_1\vec{SA}+P_2\vec{SB}+P_3\vec{SC} = \vec{0}.$$

## 2.6. Mješoviti produkt

**def:** Za vektore $\vec{a}, \vec{b}, \vec{c} \in V^3$ mješoviti produkt $(\vec{a}, \vec{b}, \vec{c})$ definira se kao

DA!
$$(\vec{a}, \vec{b}, \vec{c}) = \underbrace{(\vec{a} \times \vec{b})}_{\text{vektorski produkt}} \cdot \vec{c} \qquad \in \mathbb{R}$$

skalarni produkt

**svojstva:** • Za $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, $\vec{c} = (c_1, c_2, c_3)$ vrijedi

DA!
$$(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

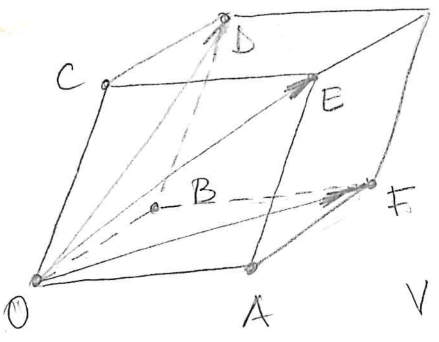
• $(\vec{a}, \vec{b}, \vec{c}) = (\vec{b}, \vec{c}, \vec{a}) = (\vec{c}, \vec{a}, \vec{b}) = -(\vec{a}, \vec{c}, \vec{b}) = -(\vec{b}, \vec{a}, \vec{c}) = -(\vec{c}, \vec{b}, \vec{a})$

• $(\vec{a}, \vec{b}, \vec{c}) = 0$ ako i samo ako su vektori $\vec{a}, \vec{b}, \vec{c}$ komplanarni

**Zad 27.:** Dokažite da plošne dijagonale koje izlaze iz jednog vrha zadanog paralelepipeda volumena $V$ razapinju novi paralelepiped volumena $2V$.

DA!

**Rj:**




$\vec{a} = \vec{OA}$, $\vec{b} = \vec{OB}$, $\vec{c} = \vec{OC}$

$\vec{d} = \vec{OD}$, $\vec{e} = \vec{OE}$, $\vec{f} = \vec{OF}$

$\vec{d} = \vec{b} + \vec{c}$, $\vec{e} = \vec{a} + \vec{c}$, $\vec{f} = \vec{a} + \vec{b}$

$V = |(\vec{a}, \vec{b}, \vec{c})|$

$V' = |(\vec{d}, \vec{e}, \vec{f})| = |(\vec{b}+\vec{c}, \vec{a}+\vec{c}, \vec{a}+\vec{b})| = |(\vec{b}+\vec{c}, \vec{a}+\vec{c}, \vec{a}) + (\vec{b}+\vec{c}, \vec{a}+\vec{c}, \vec{b})|$

$= |(\vec{b}+\vec{c}, \vec{a}, \vec{a}) + (\vec{b}+\vec{c}, \vec{c}, \vec{a}) + (\vec{b}+\vec{c}, \vec{a}, \vec{b}) + (\vec{b}+\vec{c}, \vec{c}, \vec{b})| =$

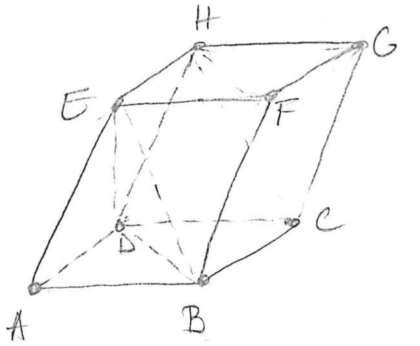
$= |(\vec{b}, \vec{c}, \vec{a}) + (\vec{c}, \vec{c}, \vec{a}) + (\vec{b}, \vec{a}, \vec{b}) + (\vec{c}, \vec{a}, \vec{b}) + (\vec{b}, \vec{c}, \vec{b}) + (\vec{c}, \vec{c}, \vec{b})|$

$= |2(\vec{a}, \vec{b}, \vec{c})| = 2\,|(\vec{a}, \vec{b}, \vec{c})|.$

**Zad 28.:** Odredite volumen pravilnog tetraedra duljine brida $a$.

NE!

4:




$$V(ABDE) = \frac{1}{3} B \cdot v = \frac{1}{3} \cdot \frac{1}{2} P_{ABCD} \cdot v =$$

$\vec{AB} = \vec{a}$ $\qquad = \frac{1}{6} V(ABCDEFGH)$

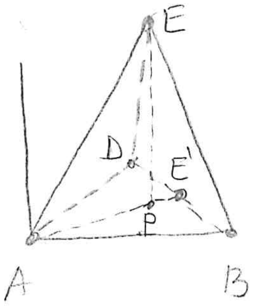
$\vec{AD} = \vec{b}$

$\vec{AE} = \vec{c}$ $\qquad = \frac{1}{6} (\vec{a}, \vec{b}, \vec{c}) =$

$$= \frac{1}{6} (\vec{a} \times \vec{b}) \cdot \vec{c} = \frac{1}{6} |\vec{a} \times \vec{b}| \cdot |\vec{c}| \cos \underbrace{\measuredangle(\vec{a} \times \vec{b}, \vec{c})}_{\varphi} =$$

$$= \frac{1}{6} |\vec{a}| \cdot |\vec{b}| \sin \measuredangle(\vec{a}, \vec{b}) \cdot |\vec{c}| \cos\varphi =$$

$$= \frac{1}{6} a^3 \sin 60° \cdot \cos\varphi = \frac{\sqrt{3}}{12} a^3 \cdot \cos\varphi$$




Neka je P projekcija točke E na ABD. Tada je $\vec{EP} \perp \vec{a}$ i $\vec{EP} \perp \vec{b}$ pa su $\vec{a} \times \vec{b}$ i $\vec{PE}$ kolinearni

$\Rightarrow \varphi = 90° - \alpha$, gdje je $\alpha = \measuredangle EAP$

$E' = AP \cap BD$. $E'$ je polovište od $\overline{BD}$. Sada je

$$\vec{AE'} \cdot \vec{AE} = |\vec{AE'}| \cdot |\vec{AE}| \cdot \cos\alpha \Rightarrow \cos\alpha = \frac{\frac{1}{2}(\vec{a} + \vec{b}) \cdot \vec{c}}{|\frac{1}{2}(\vec{a} + \vec{b})| \cdot |\vec{c}|} =$$

$$= \frac{\vec{a}\vec{c} + \vec{b}\vec{c}}{a\sqrt{3} \cdot a} = \frac{2a^2 \cos 60°}{a^2\sqrt{3}} = \frac{a^2}{a^2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

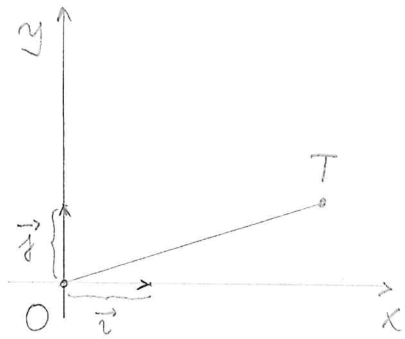
$$\Rightarrow \cos\alpha = \cos(90° - \varphi) = \sin\varphi \Rightarrow \cos\varphi = \sqrt{1 - \frac{1}{3}} = \sqrt{\frac{2}{3}}$$

$$\Rightarrow V = \frac{\sqrt{3}}{12} a^3 \cdot \frac{\sqrt{2}}{\sqrt{3}} = \frac{a^3\sqrt{2}}{12}$$

# 3. ANALITIČKA GEOMETRIJA RAVNINE I PROSTORA

## 3.1. Koordinatni sustav i koordinate točaka

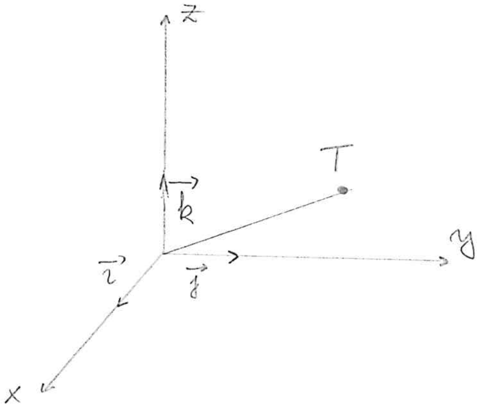
NE!




**Koordinatni sustav u ravnini**

$(O, \vec{i}, \vec{j})$ $\quad \vec{i}=(1,0), \quad \vec{j}=(0,1)$

$T=(x,y)$ $\quad \vec{OT}=x\vec{i}+y\vec{j}$




**Koordinatni sustav u prostoru**

$(O, \vec{i}, \vec{j}, \vec{k})$ $\quad \vec{i}=(1,0,0), \ \vec{j}=(0,1,0), \vec{k}=(0,0,1)$

$\vec{OT}=x\vec{i}+y\vec{j}+z\vec{k}$ $\quad T=(x,y,z)$

Udaljenost točaka A i B $\quad d(A,B)=|\vec{AB}|$

Npr: Udaljenost točaka $A=(1,0)$, $B=(3,7)$ $\quad d(A,B)=\sqrt{(3-1)^2+(7-0)^2}=\sqrt{53}$

NE! Udaljenost točaka $C=(1,-1,2)$, $D=(0,1,3)$ je

$$d(C,D)=\sqrt{(1-0)^2+(-1-1)^2+(2-3)^2}=\sqrt{1+4+1}=\sqrt{6}$$

zad 1.: Odredite koordinate točke T za koje vrijedi $\vec{AT}=\frac{1}{3}\vec{AB}$

NE! pri čemu je $A=(1,7,-3)$, $B=(-2,-2,-2)$.

rj: $T=(x,y,z)$ $\quad \vec{AT}=(x-1,y-7,z+3)$, $\vec{AB}=(-3,-9,1)$, $\vec{AT}=\frac{1}{3}\vec{AB}$

$$\Rightarrow (x-1,y-7,z+3)=\left(-1,-3,\frac{1}{3}\right) \Rightarrow \begin{cases} x-1=-1 & \Rightarrow \boxed{x=0} \\ y-7=-3 & \Rightarrow \boxed{y=4} \\ z+3=\frac{1}{3} & \Rightarrow \boxed{z=-\frac{8}{3}} \end{cases}$$

$$\Rightarrow \boxed{T=\left(0,4,-\frac{8}{3}\right)}$$

**Zad 2.:** Odredite u kojem omjeru točka $T_i$ dijeli dužinu $\overline{AB}$,

NE! gdje je $A=(3,-5)$, $B=(-9,1)$:

a) $T_1=(-1,-3)$, b) $T_2=(9,-8)$, c) $T_3=(-7,-1)$

Rj: Općenito za $A=(a_1,a_2)$, $B=(b_1,b_2)$, $T=(x,y)$

$$\overrightarrow{AT}=\lambda\overrightarrow{AB} \Rightarrow (x-a_1,y-a_2)=\lambda(b_1-a_1,b_2-a_2)$$

$$\Rightarrow \lambda=\frac{x-a_1}{b_1-a_1}=\frac{y-a_2}{b_2-a_2}$$

a) $\frac{x-a_1}{b_1-a_1}=\frac{-1-3}{-9-3}=\frac{1}{3}$, $\frac{y-a_2}{b_2-a_2}=\frac{-3+5}{1+5}=\frac{1}{3} \Rightarrow \overrightarrow{AT_1}=\frac{1}{3}\overrightarrow{AB}$

b) $\frac{x-a_1}{b_1-a_1}=\frac{9-3}{-9-3}=-\frac{1}{2}$, $\frac{y-a_2}{b_2-a_2}=\frac{-8+5}{1+5}=-\frac{1}{2} \Rightarrow \overrightarrow{AT_2}=-\frac{1}{2}\overrightarrow{AB}$

c) $\frac{x-a_1}{b_1-a_1}=\frac{-7-3}{-9-3}=\frac{5}{6}$, $\frac{y-a_2}{b_2-a_2}=\frac{-1+5}{1+5}=\frac{4}{6}$.

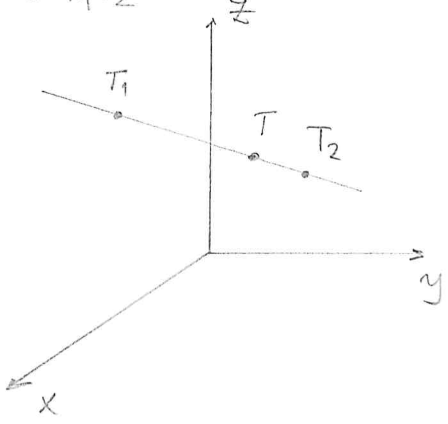
$$\frac{x-a_1}{b_1-a_1}\neq\frac{y-a_2}{b_2-a_2} \Rightarrow \text{Točka } T_3 \text{ nije na pravcu } AB.$$

## 3.2. Pravac

DA! Aksiomi pravca

- Svake dvije različite točke pripadaju jednom i samo jednom pravcu
- Postoje tri nekolinearne točke (postoji ravnina)

Pravac $p$ je u prostoru zadan s dvije različite točke $T_1$ i $T_2$. Za svaku točku $T$ koja leži na pravcu $p$ vektori $\overrightarrow{T_1T_2}$ i $\overrightarrow{T_1T}$ su kolinearni, tj. $\exists\, t\in\mathbb{R}$ t.d. $\overrightarrow{T_1T}=t\,\overrightarrow{T_1T_2}$




$\overrightarrow{T_1T_2}=(a,b,c)$, $T_1=(x_1,y_1,z_1)$, $T=(x,y,z)$

$$(x-x_1,y-y_1,z-z_1)=t(a,b,c)$$

$$\Rightarrow \left.\begin{aligned} x&=x_1+at\\ y&=y_1+bt\\ z&=z_1+ct\end{aligned}\right\}\ \text{parametarska jednadžba pravca}$$

Eliminacijom parametra t dobivamo kanonsku jednadžbu pravca

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}, \text{ tj. } \frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Primjer: Jednadžba x-osi glasi $\begin{cases} x=t \\ y=0, \quad t\in\mathbb{R} \\ z=0 \end{cases}$, tj. $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$

Pravac u ravnini (?) NE!

Promatramo ga kao pravac u prostoru koji leži u x-y ravnini (svi izrazi uz z nestaju)

Implicitna jednadžba pravca $Ax+By+C=0$

Eksplicitna jednadžba pravca $y=ax+b$

Segmentna jednadžba pravca $\frac{x}{m}+\frac{y}{n}=1$

Jednadžba pravca kroz dvije točke $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

Ako su $T_1$ i $T_2$ dvije različite točke pravca p, onda je $\overrightarrow{T_1T_2}$ vektor smjera pravca p (možemo uzeti i bilo koji drugi vektor kolinearan s $\overrightarrow{T_1T_2}$).

Jednadžba normale (pravca okomitog na p koji prolazi kroz $(x_0, y_0)$)

$$y-y_0=-\frac{1}{a}(x-x_0) \qquad (\text{ako je } p \dots y=ax+b)$$

Uvjet paralelnosti dvaju pravaca: Pravci su paralelni ako i samo ako su im koeficijenti smjera jednaki.

Uvjet okomitosti dvaju pravaca: Pravci su okomiti ako i samo ako su im koeficijenti smjera suprotnih predznaka i recipročni.

Kut između dva pravca: Ako su $k_1$ i $k_2$ koeficijenti smjera pravaca $p_1$ i $p_2$, onda je $\operatorname{tg}\sphericalangle(p_1,p_2)=\frac{k_2-k_1}{1+k_1k_2}$ IZVESTI!

**zad 3.:** Pravac p je zadan parametarskom jednadžbom

NE!
$$\begin{cases} x = 2+3t \\ y = -1+2t \end{cases}$$

a) Odredite neke tri točke na pravcu p.

b) Odredite vektor smjera i vektor normale

c) U kojim točkama pravac siječe koordinatne osi

d) Napišite implicitnu i eksplicitnu jednadžbu pravca p.

e) Koji kut zatvara pravac p s osi x?

**rj:** a) za $t=0 \ldots (2,-1) \in p$, za $t=1 \ldots (5,1) \in p$,

za $t=5 \ldots (17,9) \in p$.

b) Vektor smjera: $T_1=(2,-1)$, $T_2=(5,1) \in p \Rightarrow \overrightarrow{T_1T_2}=(3,2) \longrightarrow$ vektor smjera

Vektor normale: $\frac{y+1}{2} = \frac{x-2}{3} \Rightarrow y = \frac{2}{3}(x-2)-1$

Jednadžba normale: $y-y_0 = -\frac{3}{2}(x-x_0)$, $(x_0,y_0)\in p$

$$y+1 = -\frac{3}{2}(x-2)$$

$T_1=(2,-1)$, $T_3=(4,-4)$ $\quad \overrightarrow{T_1T_3}=(2,-3) \longrightarrow$ vektor normale

c) $x=0 \Longleftrightarrow t=-\frac{2}{3} \Longleftrightarrow y=-1-\frac{4}{3}=-\frac{7}{3}$

$\longrightarrow$ p siječe os y u točki $(0,-\frac{7}{3})$

$y=0 \Longleftrightarrow t=\frac{1}{2} \Longleftrightarrow x=2+\frac{3}{2}=\frac{7}{2}$

$\longrightarrow$ p siječe os x u točki $(\frac{7}{2},0)$

d) Eksplicitna: $y=\frac{2}{3}x-\frac{7}{3}$

Implicitna: $2x-3y-7=0$

e) $p \ldots y=\frac{2}{3}x-\frac{7}{3}$ $\quad k_1=\frac{2}{3}$

x-os $\ldots y=0$ $\quad k_2=0$

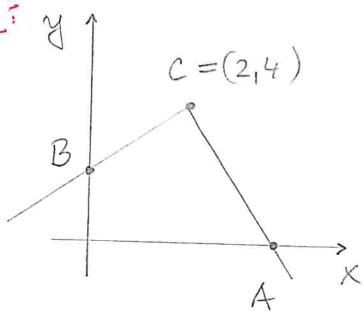
$$\operatorname{tg}\varphi = \frac{\frac{2}{3}-0}{1+0} = \frac{2}{3}$$

$$\boxed{\varphi = \operatorname{arctg}\frac{2}{3}}$$

**Zad 4.:** Dana je točka $C=(2,4)$. Točkom $C$ prolaze dva međusobno okomita pravca. Prvi siječe os $x$ u točki $A$, drugi siječe os $y$ u točki $B$. Odredite geometrijsko mjesto polovišta dužine $\overline{AB}$, ako pravci rotiraju oko točke $C$, ostajući međusobno okomiti.

DA!

**Rj:**




$$P_1 \ldots\ y-4=k(x-2) \qquad P_1=CA,\ P_2=CB$$

$$P_2 \ldots\ y-4=-\frac{1}{k}(x-2)$$

$$A=P_1\cap x\text{-os} \qquad A=(x_A,0)$$

$$A\in P_1 \Rightarrow -4=k(x_A-2) \Rightarrow x_A=\frac{2k-4}{k}$$

$$B=P_2\cap y\text{-os} \qquad B=(0,y_B)$$

$$B\in P_2 \Rightarrow y_B-4=-\frac{1}{k}(-2) \Rightarrow y_B=\frac{2}{k}+4$$

$$A=\left(2-\frac{4}{k},0\right),\ B=\left(0,\frac{2}{k}+4\right) \Rightarrow P_{AB}=\left(1-\frac{2}{k},\frac{1}{k}+2\right)$$

$$x_P=1-\frac{2}{k} \Rightarrow \frac{2}{k}=1-x_P \Rightarrow \frac{1}{k}=\frac{1}{2}-\frac{x_P}{2} \Rightarrow y_P=\frac{5}{2}-\frac{x_P}{2}$$

Geometrijsko mjesto polovišta dužine $\overline{AB}$ je pravac $y=-\frac{1}{2}x+\frac{5}{2}$.

**Zad 5.:** Odredite sjecište pravaca $\frac{x+1}{1}=\frac{y}{1}=\frac{z-1}{2}$, $\frac{x}{1}=\frac{y+1}{3}=\frac{z-2}{4}$.

NE!

**Rj:** 1. način:

$$P_1 \ldots \begin{cases} x=-1+t \\ y=t \\ z=1+2t \end{cases} \qquad P_2 \ldots \begin{cases} x=s \\ y=-1+3s \\ z=2+4s \end{cases}$$

Točku presjeka tražimo rješavanjem sustava
$$\begin{cases} -1+t=s & (1) \\ t=-1+3s & (2) \\ 1+2t=2+4s & (3) \end{cases}$$

$$(2)\ \&\ (1) \Rightarrow -1-1+3s=s \Rightarrow -2=-2s \Rightarrow \boxed{s=1} \Rightarrow \boxed{t=2}$$

$s=1$ i $t=2$ ne zadovoljava jednadžbu (3) jer je $1+2\cdot 2\neq 2+4\cdot 1$

$\Rightarrow$ Pravci $P_1$ i $P_2$ se ne sijeku.

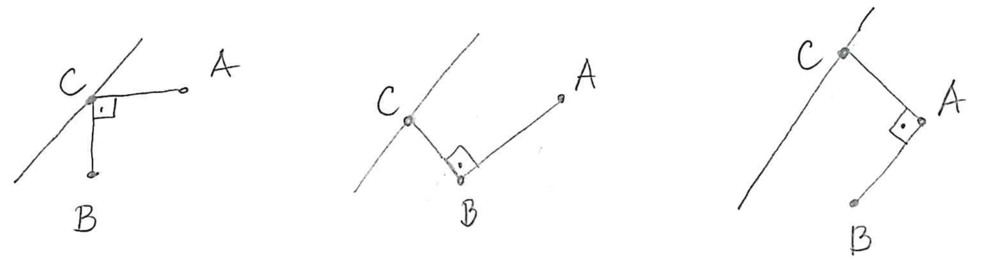
2. način: Tražimo rješenje sustava $\left\{\begin{array}{l}\begin{cases} x+1=y \\ 2y=z-1 \end{cases} \\ \begin{cases} 3x=y+1 \\ 4y+4=3z-6 \end{cases}\end{array}\right.$

(taj sustav neće imati rješenje).

**Zad 6.:** Na pravcu $\frac{x+2}{-1}=\frac{y+3}{-1}=\frac{z-3}{1}$ odredite sve točke koje s točkama $A=(-2,1,1)$ i $B=(0,-7,4)$ čine pravokutan trokut.

DA! ✓

rj:




Pravi kut je ili u vrhu A ili u vrhu B ili na pravcu.

Točke pravca su oblika $C=(-2-t,-3-t,3+t)$.

$\vec{AB}=(2,-8,3)$, $\vec{AC}=(-t,-4-t,2+t)$, $\vec{BC}=(-2-t,4-t,-1+t)$

1.) Pravi kut u vrhu $A \Longleftrightarrow \vec{AB}\perp\vec{AC} \Longleftrightarrow \vec{AB}\cdot\vec{AC}=0$

$\Longleftrightarrow -2t+32+8t+6+3t=0 \Longleftrightarrow 38+9t=0 \Longleftrightarrow t=-\frac{38}{9}$

$\Longleftrightarrow \boxed{C=\left(\frac{20}{9},\frac{11}{9},\frac{-11}{9}\right)}$

2.) Pravi kut u vrhu $B \Longleftrightarrow \vec{AB}\perp\vec{BC} \Longleftrightarrow \vec{AB}\cdot\vec{BC}=0$

$\Longleftrightarrow -4-2t-32+8t-3+3t=0 \Longleftrightarrow 9t-39=0 \Longleftrightarrow t=\frac{13}{3}$

$\Longleftrightarrow \boxed{C=\left(\frac{-19}{3},\frac{-22}{3},\frac{22}{3}\right)}$

3.) Pravi kut u vrhu $C \Longleftrightarrow \vec{AC}\perp\vec{BC} \Longleftrightarrow \vec{AC}\cdot\vec{BC}=0$

$\Longleftrightarrow 2t+t^2+t^2-16+t^2+t-2=0 \Longleftrightarrow 3t^2+3t-18=0$

$\Longleftrightarrow t^2+t-6=0 \Longleftrightarrow t\in\{2,-3\}$

$\Longleftrightarrow \boxed{C\in\{(-4,-5,5),(1,0,0)\}}$

pitaj ih koliko rješenja očekuju u ovom slučaju.

**Zad 7.:** Odredite jednadžbu pravca koji prolazi ishodištem

DA! ✓ i siječe pravce

$$\frac{x-\frac{7}{2}}{5}=\frac{y}{3}=\frac{z-\frac{5}{2}}{1} \quad i \quad \frac{x+3}{1}=\frac{y-12}{1}=\frac{z+9}{1}=s$$

Rj: Neka traženi pravac $p$ siječe pravac $p_1$ u točki $A=\left(\frac{7}{2}+5t, 3t, \frac{5}{2}+t\right)$, a pravac $p_2$ u točki $B=(-3+s, 12+s, -9+s)$.

Pravcu $p$ pripada i točka $O=(0,0,0)$, a to znači da postoji $\lambda \in \mathbb{R}$ t.d. $\overrightarrow{OA}=\lambda\overrightarrow{OB}$.

$$\begin{cases} \frac{7}{2}+5t=-3\lambda+\lambda s & (1)\\ 3t=12\lambda+\lambda s & (2)\\ \frac{5}{2}+t=-9\lambda+\lambda s & (3)\end{cases}$$

$(1)-(2):\quad \frac{7}{2}+2t=-15\lambda \quad /\cdot 2$

$(1)-(3):\quad 1+4t=6\lambda \quad /\cdot(-1)$ $\quad \oplus$

$$6=-36\lambda \Rightarrow \boxed{\lambda=-\frac{1}{6}}$$

$$\Rightarrow 1+4t=-1 \Rightarrow \boxed{t=-\frac{1}{2}}$$

$$(2) \Rightarrow -\frac{3}{2}=-2-\frac{1}{6}s \Rightarrow \boxed{s=-3}$$

$$\Rightarrow A=\left(1,-\frac{3}{2},2\right),\quad B=(-6,9,-12)$$

Traženi pravac je $p \ldots \ \frac{x}{1}=\frac{y}{-\frac{3}{2}}=\frac{z}{2}$.

## 3.3. Ravnina

DA! Ravnina $\Pi$ u prostoru je zadana s tri točke $T_1, T_2$ i $T_3$ koje ne leže na istom pravcu. Za svaku točku $T$ koja leži u ravnini $\Pi$ vektori $\overrightarrow{T_1T}$, $\overrightarrow{T_1T_2}$ i $\overrightarrow{T_1T_3}$ su komplanarni, tj. volumen paralelepipeda kojeg razapinju tri vektori jednak je 0, odnosno

$$\overrightarrow{T_1T}\cdot\left(\overrightarrow{T_1T_2}\times\overrightarrow{T_1T_3}\right)=0 \qquad (*)$$

$\vec{r_1}=\overrightarrow{OT_1}$, $\vec{r}=\overrightarrow{OT}$, $\vec{n}=\overrightarrow{T_1T_2}\times\overrightarrow{T_1T_3}$ $\Rightarrow$ $(\vec{r}-\vec{r_1})\cdot\vec{n}=0$ vektorska jednadžba ravnine

$\vec{n}=$ vektor normale $\qquad$ za $\vec{n}=(A,B,C)$, $\vec{r_1}=(x_1,y_1,z_1)$, $\vec{r}=(x,y,z)$

dobivamo jednadžbu ravnine kroz točku $T_1=(x_1,y_1,z_1)$

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$

– opći oblik jednadžbe ravnine $Ax+By+Cz+D=0$ (implicitna)

Ako je $T=(x,y,z)$, $T_i=(x_i,y_i,z_i)$, $i=1,2,3$, onda jednadžbu ravnine možemo zapisati u obliku:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

Ako ravnina $\Pi$ ne prolazi ishodištem i ako za točke $T_1, T_2$ i $T_3$ odaberemo sjecišta ravnine s koordinatnim osima,

$$T_1=(a,0,0),\ T_2=(0,b,0),\ T_3=(0,0,c),\ a,b,c\neq 0,$$

onda rješavanjem gornje determinante dobijemo segmentni oblik jednadžbe ravnine $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, $a,b,c\neq 0$

Parametarska jednadžba ravnine kroz točku $(x_0,y_0,z_0)$ razapeta (nekolinearnim) vektorima $\vec{a}=(a_x,a_y,a_z)$ i $\vec{b}=(b_x,b_y,b_z)$ glasi:

$$\begin{cases} x=x_0+ta_x+sb_x \\ y=y_0+ta_y+sb_y \\ z=z_0+ta_z+sb_z \end{cases}$$

Zad 8.: Napišite jednadžbu ravnine kroz točke $A=(1,0,0)$, $B=(1,1,1)$,
NE! $C=(0,0,-1)$.

R:
$$\begin{vmatrix} x-1 & y-0 & z-0 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{vmatrix}=0 \implies (x-1)(-1)+y\cdot(-1)-z\cdot(-1)=0$$
$$\implies -x+1-y+z=0$$
$$\implies -x-y+z+1=0$$

**zad 9.:** Napišite implicitnu jednadžbu ravnine

NE!

$$\begin{cases} x = 1+t+s \\ y = t-s \\ z = 2-t+2s \end{cases} \quad t, s \in \mathbb{R}$$

**rj:** 1. način: Odredimo neke tri točke koje pripadaju ravnini npr za $t=0, s=0$, za $t=1, s=0$ i $t=0, s=1$ i onda raspišemo determinantu. $A=(1,0,2)$, $B=(2,1,1)$, $C=(2,-1,4)$.

2. način: Eliminacija parametara:

$$\left.\begin{array}{l} x-y = 1+2s \\ z+y = 2+s \;/\cdot(-2) \end{array}\right\} \oplus \implies x-3y-2z=-3$$

**zad 10.:** Odredite sjecište pravca $\frac{x-1}{2} = \frac{y-0}{1} = \frac{z+2}{2}$ i ravnine

~~DA~~ / NE

$x-y+4z-5=0$

**rj:**

$$p \ldots \begin{cases} x = 1+2t \\ y = t \\ z = -2+2t \end{cases} \qquad \text{Za } (x_0, y_0, z_0) = p \cap \pi \text{ vrijedi}$$

$$(1+2t)-t+4(-2+2t)-5=0 \implies 9t-12=0$$

$$\implies t=\frac{4}{3} \implies p \cap \pi = \left\{\left(\frac{11}{3}, \frac{4}{3}, \frac{2}{3}\right)\right\}$$

**zad 11.:** Odredite $D \in \mathbb{R}$ tako da pravac

DA!

$$\begin{cases} x-y+z+1=0 \\ 2x-3y-z+D=0 \end{cases}$$ siječe os $z$.

**rj:** Ovaj pravac je zadan kao presjek dvije ravnina $\Pi_1$ i $\Pi_2$. Prva ravnina siječe os $z$ u točki $(0,0,-1)$. $p=\Pi_1 \cap \Pi_2$ siječe os $z$ u toj točki $\implies (0,0,-1) \in \Pi_2 \implies 2\cdot 0 - 3\cdot 0 + 1 + D = 0 \implies \boxed{D=-1}$

**zad 12.:** Odredite $\lambda \in \mathbb{R}$ t.d. se ravnine

DA! ✓

$$\Pi_1 \ldots x-y+z=0$$
$$\Pi_2 \ldots 3x-y-z+2=0$$
$$\Pi_3 \ldots 4x-y-2z+\lambda=0$$

sijeku po istom pravcu.

rj: $\Pi_1 \cap \Pi_2 \ldots \begin{cases} x-y+z=0 \\ 3x-y-z+2=0 \end{cases} \Big/ \ominus \implies 2x-2z+2=0$

$\implies x = z-1 \implies y = x+z = 2z-1$

$p = \Pi_1 \cap \Pi_2 \ldots \quad x = t-1, \; z = t, \; y = 2t-1 \qquad t \in \mathbb{R}$

$p \in \Pi_3 \implies \forall t \in \mathbb{R}$ je $4(t-1) - (2t-1) - 2t + \lambda = 0$

$\implies \cancel{4t} - 4 - \cancel{2t} + 1 - \cancel{2t} + \lambda = 0 \quad \forall t \in \mathbb{R}$

$\implies \boxed{\lambda = 3}$

Zad 13.: Odredite pravac $p$ koji je paralelan s ravninama

DA! ✓ $\Pi_1, \Pi_2$ i siječe pravce $q_1, q_2$, gdje je:

$\Pi_1 \ldots \; 3x + 12y - 3z - 5 = 0 \qquad q_1 \ldots \; \frac{x+5}{2} = \frac{y-3}{-4} = \frac{z+1}{3}$

$\Pi_2 \ldots \; 3x - 4y + 9z + 7 = 0 \qquad q_2 \ldots \; \frac{x-3}{-2} = \frac{y+1}{3} = \frac{z-2}{4}$

rj: $\left.\begin{array}{l} p \parallel \Pi_1 \implies \text{vektor smjera } \vec{s} \perp \vec{n_1} = (3,12,-3) \\ p \parallel \Pi_2 \implies \text{vektor smjera } \vec{s} \perp \vec{n_2} = (3,-4,9) \end{array}\right\} \iff \vec{s} \parallel \vec{n_1} \times \vec{n_2}$

$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & -1 \\ 3 & -4 & 9 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 1 & 4 \\ 3 & -4 \end{matrix} = 36\vec{i} - 3\vec{j} - 4\vec{k} - 9\vec{j} - 4\vec{i} - 12\vec{k} =$

$= 32\vec{i} - 12\vec{j} - 16\vec{k} \implies \vec{n_1} \times \vec{n_2} = 4(8,-3,-4)$

Možemo uzeti $\vec{s} = (8,-3,-4)$ (vektor smjera pravca $p$).

$p \cap q_1 \neq \emptyset \implies p$ i $q_1$ leže u istoj ravnini $M_1$

$p \cap q_2 \neq \emptyset \implies p$ i $q_2$ –||– $M_2$

$M_1 \ldots$ određena točkom $(-5,3,-1) \in q_1$ i vektorima smjera

$(2,-4,3), \; (8,-3,-4) \qquad \begin{vmatrix} x+5 & y-3 & z+1 \\ 2 & -4 & 3 \\ 8 & -3 & -4 \end{vmatrix} = 0$

$\implies M_1 \ldots \; 25x + 32y + 26z + 55 = 0$

$M_2 \ldots$ određena točkom $(3,-1,2) \in q_2$ i vektorima $(-2,3,4)$,

$(8,-3,-4) \qquad \begin{vmatrix} x-3 & y+1 & z-2 \\ -2 & 3 & 4 \\ 8 & -3 & -4 \end{vmatrix} \implies \ldots \; M_2 \ldots \; 4y - 3z + 10 = 0$

$p = M_1 \cap M_2$

Zad 14.: Napišite kanonski oblik jednadžbe pravca koji je paralelan s ravninom $x+2y+3z=8$, leži u ravnini $2x-y+z=3$ i prolazi točkom $(1,2,3)$.

DA!

Rj: $\Pi_1 \ldots x+2y+3z=8$  $\Pi_3$ = ravnina paralelna s $\Pi_1$ u kojoj leži pravac p.

$\Pi_2 \ldots 2x-y+z=3$

$\Pi_1 \parallel \Pi_3 \Rightarrow$ te dvije ravnine imaju kolinearne vektore normale pa je jednadžba ravnine $\Pi_3$ $x+2y+3z=D$. No, $(1,2,3) \in p \subseteq \Pi_3$ pa je $1+2\cdot 2+3\cdot 3=D \Rightarrow D=14 \Rightarrow \Pi_3 \ldots x+2y+3z=14$

$p=\Pi_2 \cap \Pi_3$ Za točke pravca p vrijedi $y=2x+z-3=\frac{14-x-3z}{2}$

$\Rightarrow 4x+2z-6=14-x-3z \Rightarrow 5x+5z=20 \Rightarrow z=4-x,\ y=2x+4-x-3=$
$=x+1$

$p \ldots \frac{x}{1}=\frac{y-1}{1}=\frac{z-4}{-1}$ Točka $(1,2,3)$ zaista pripada tom pravcu.

Zad 15.: (DZ) Odredite jednadžbu ravnine paralelne s vektorom $\vec{s}=(2,1,-1)$ koja os x siječe u točki $x=3$, a os y u točki $y=-2$.

DA!

Rj: Segmentni oblik $\Pi \ldots \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$(3,0,0) \in \Pi \Rightarrow a=3$, $(0,-2,0) \in \Pi \Rightarrow b=-2$

$\Pi \ldots \frac{1}{3}x-\frac{1}{2}y+\frac{1}{c}z-1=0$ vektor normale $\vec{n}=\left(\frac{1}{3},-\frac{1}{2},\frac{1}{c}\right)$

$\Pi \parallel \vec{s} \Leftrightarrow \vec{n}\cdot\vec{s}=0 \Leftrightarrow \frac{2}{3}-\frac{1}{2}-\frac{1}{c}=0 \Leftrightarrow \frac{1}{c}=\frac{1}{6} \Leftrightarrow \boxed{c=6}$

Jednadžba ravnine glasi $\frac{x}{3}+\frac{y}{-2}+\frac{z}{6}=1$.

Zad 16.: Odredite zajedničku normalu pravaca $\frac{x}{1}=\frac{y}{1}=\frac{z}{0}$ i $\frac{x-1}{0}=\frac{y-2}{1}=\frac{z-3}{0}$.

DA!

Rj: Koeficijenti smjera pravaca $p_1$ i $p_2$ su redom $\vec{s}_1=(1,1,0)$ i $\vec{s}_2=(0,1,0)$. Kako su $\vec{s}_1$ i $\vec{s}_2$ linearno nezavisni, $p_1 \nparallel p_2$. Nadalje, za točke pravca $p_1$ vrijedi $z=0$, a za točke pravca $p_2$ vrijedi $z=3$ pa je $p_1 \cap p_2 = \emptyset$. Dakle, $p_1$ i $p_2$ su mimoilazni.

$p$ = zajednička normala = pravac okomit na oba pravca koji siječe oba pravca.

$p \perp p_1$ & $p \perp p_2$ $\Rightarrow$ vektor smjera $\vec{s}$ pravca $p$ je okomit na $\vec{s}_1$ i $\vec{s}_2$. $\Rightarrow \vec{s} = \vec{s}_1 \times \vec{s}_2$

$$\vec{s} = \left|\begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{array}\right| = \vec{k} = (0,0,1)$$

$$p \ldots \frac{x-x_0}{0} = \frac{y-y_0}{0} = \frac{z-z_0}{1} \qquad p \ldots \begin{cases} x = x_0 \\ y = y_0 \\ z = t + z_0 \end{cases}, \ t \in \mathbb{R}$$

$$p \cap p_1 \neq \emptyset \Rightarrow \exists\, t \in \mathbb{R} \text{ t.d. } \frac{x_0}{1} = \frac{y_0}{1} = \frac{t+z_0}{0} \Rightarrow \boxed{x_0 = y_0},\ t = -z_0$$

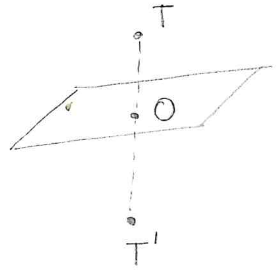
$$p \cap p_2 \neq \emptyset \Rightarrow \exists\, s \in \mathbb{R} \text{ t.d. } \frac{x_0-1}{0} = \frac{y_0-2}{1} = \frac{s+z_0-3}{0} \Rightarrow \boxed{x_0-1=0},\ -s = z_0 - 3$$

$$\Rightarrow p \ldots \begin{cases} x = 1 \\ y = 1 \\ z = u \end{cases}, \ u \in \mathbb{R} \qquad u = t + z_0$$

**Zad 17.:** Odredite ortogonalnu projekciju točke $T=(2,3,1)$ na ravninu $\pi \ldots x+y-z-7=0$ te simetričnu točku točki $T$ s obzirom na ravninu $\pi$.

DA!!

Rj: vektor normale ravnine $\pi$ je $\vec{n} = (1,1,-1)$




$O$ = ortogonalna projekcija točke $T$ na $\pi$.
Koeficijent smjera pravca $OT$ je $(1,1,-1)$ i $T \in OT$ pa je $OT \ldots \frac{x-2}{1} = \frac{y-3}{1} = \frac{z-1}{-1}$

$$OT \ldots \begin{cases} x = 2+t \\ y = 3+t \\ z = 1-t \end{cases}, \ t \in \mathbb{R}$$

$O = OT \cap \pi$ Tražimo $t \in \mathbb{R}$ t.d. $(2+t, 3+t, 1-t) \in \pi$.

$$2+t+3+t-1+t-7=0 \Rightarrow 3t=3 \Rightarrow t=1 \Rightarrow \boxed{O=(3,4,0)}$$

$O$ je polovište dužine $\overline{TT'}$ pa vrijedi

$$(3,4,0)=\left(\frac{x_T+x_{T'}}{2},\frac{y_T+y_{T'}}{2},\frac{z_T+z_{T'}}{2}\right)=\left(\frac{2+x_{T'}}{2},\frac{3+y_{T'}}{2},\frac{1+z_{T'}}{2}\right)$$

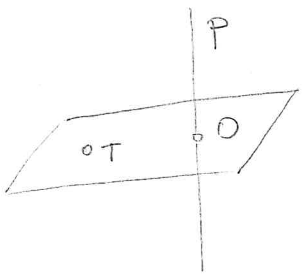
$\Rightarrow \boxed{T'=(4,5,-1)}$

Zad 18.: Odredite ortogonalnu projekciju točke $T=(2,3,1)$ na pravac

DA! ?

$$p \dots \begin{cases} x=t-2 \\ y=2t-1 \\ z=-2t+4 \end{cases}$$

rj:



$\Pi=$ ravnina okomita na $p$ koja prolazi točkom $T$.

Njen vektor normale je vektor smjera pravca $p$, tj. $(1,2,-2)$.

$\Pi \dots\ x+2y-2z=D$, $\quad T\in\Pi \Rightarrow 2+6-2=D$
$\Rightarrow D=6$

$\Rightarrow \Pi \dots\ x+2y-2z=6$

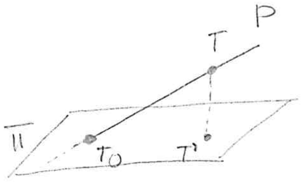
$O=$ ortogonalna projekcija točke $T$ na pravac $p$. Tražimo tak $t$ t.d. $(t-2,2t-1,-2t+4)\in\Pi$, $\quad t-2+4t-2+4t-8=6 \Rightarrow 9t=18$
$\Rightarrow t=2$

$\Rightarrow \boxed{O=(0,3,0)}$

Zad 19.: Odredite ortogonalnu projekciju pravca $p \dots \begin{cases} x-y+z=1 \\ x+y+z=3 \end{cases}$

DA! !! na ravninu $\Pi \dots\ 2x+2y+z=5$.

rj:



$T_0=\Pi\cap p$. Koordinate točke $T_0$ zadovoljavaju sustav $\begin{cases} x-y+z=1 \\ x+y+z=3 \\ 2x+2y+z=5 \end{cases} \Rightarrow T_0=(1,1,1)$.

Odaberemo točku $T\neq T_0$ na $p$ npr. za $x=0$: $-y_T+z_T=1$, $y_T+z_T=3 \Rightarrow z_T=2$, $y_T=1$. $T=(0,1,2)$

$T'=$ ortogonalna projekcija točke $T$ na ravninu $\Pi$.

Vektor normale ravnine $\pi$ je $(2,2,1)$ a to je ujedno vektor smjera pravca $TT'$ $\Rightarrow$ $TT' \ldots \frac{x}{2} = \frac{y-1}{2} = \frac{z-2}{1}$ $\quad \begin{cases} x = 2t \\ y = 1+2t \\ z = 2+t \end{cases}, t \in \mathbb{R}$

$T' = \pi \cap TT' \qquad 2\cdot 2t + 2\cdot(1+2t) + (2+t) = 5 \Rightarrow 9t = 1 \Rightarrow t = \frac{1}{9}$

$T' = \left(\frac{2}{9}, \frac{11}{9}, \frac{19}{9}\right)$

ortogonalna projekcija $= T_0T' \ldots \frac{x-1}{-\frac{7}{9}} = \frac{y-1}{\frac{2}{9}} = \frac{z-1}{\frac{10}{9}}$

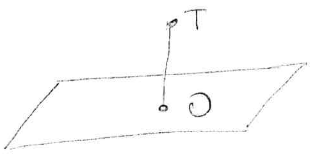
**Zad 20.:** Odredite točke jednako udaljene od ravnina

DZ

$\pi_1 \ldots 16x - 12y + 15z - 9 = 0$

$\pi_2 \ldots 12x + 9y - 20z - 19 = 0$

**Uj:** Udaljenost točke $T = (x_0, y_0, z_0)$ od ravnine $\pi \ldots Ax + By + Cz + D = 0$




$O$ = ortogonalna projekcija točke $T$ na $\pi$.

$\vec{n} = (A,B,C)$ = vektor normale ravnine $\pi$ = = vektor smjera pravca $OT$.

$OT \ldots \frac{x-x_0}{A} = \frac{y-y_0}{B} = \frac{z-z_0}{C} \qquad O = OT \cap \pi \qquad d(T,\pi) = d(O,T)$

Za DZ izvedite formulu $d(T,\pi) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2+B^2+C^2}}$

Tražimo točke $T$ takve da je $d(T,\pi_1) = d(T,\pi_2)$, tj.

$$\frac{|16x - 12y + 15z - 9|}{\underbrace{\sqrt{16^2+12^2+15^2}}_{25}} = \frac{|12x + 9y - 20z - 19|}{\underbrace{\sqrt{12^2+9^2+20^2}}_{25}}$$

$\Leftrightarrow \; 16x - 12y + 15z - 9 = 12x + 9y - 20z - 19$ ili $16x - 12y + 15z - 9 = -12x - 9y + 20z + 19$

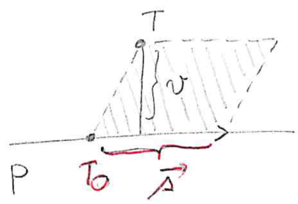
$\Leftrightarrow \; \underbrace{4x - 21y + 35z + 10 = 0}_{\Sigma_1}$ ili $\underbrace{28x - 3y - 5z - 28 = 0}_{\Sigma_2}$

Točke $T$ za koje je $d(T,\pi_1) = d(T,\pi_2)$ su točke ravnina $\Sigma_1 \cup \Sigma_2$. Uočite, $\Sigma_1$ i $\Sigma_2$ su okomite (njihovi vektori normale su okomiti)

**Zad 21.:** Odredite ~~najkraću~~ udaljenost točke $T=(2,3,1)$ od pravca $\frac{x+2}{1}=\frac{y+1}{2}=\frac{z-4}{-2}$.

DA!

**Rj:** općenito:




$\vec{OT}=\vec{r_1}$, $\vec{OT_0}=\vec{r_0}$, $T_0$ neka točka pravca, $\vec{s}$ = vektor smjera pravca p.

$$P_{paralelograma}=|\vec{TT_0}\times\vec{s}|=|\vec{s}|\cdot v \Rightarrow d(T,p)=v=\frac{|(\vec{r_0}-\vec{r_1})\times\vec{s}|}{|\vec{s}|}$$

$\vec{TT_0}=\vec{TO}+\vec{OT_0}=\vec{r_0}-\vec{r}$

U našem primjeru $\vec{s}=(1,2,-2)$, $T_0=(-2,-1,4)$

$\vec{r_0}-\vec{r_1}=(-4,-4,3)$ $\quad (\vec{r_0}-\vec{r_1})\times\vec{s}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\-4&-4&3\\1&2&-2\end{vmatrix}=\ldots=(2,-5,-4)$

$$\Rightarrow d(T,p)=\frac{\sqrt{4+25+16}}{\sqrt{1+4+4}}=\frac{\sqrt{45}}{3}=\sqrt{5}$$

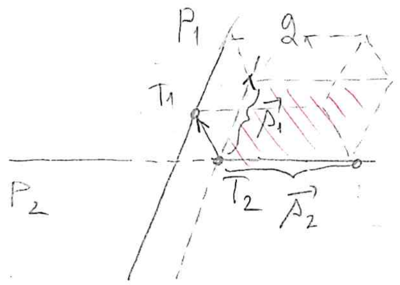
**Zad 22.:** Nađite udaljenost između pravaca

DA!

$p_1 \ldots \frac{x+5}{2}=\frac{y-3}{-4}=\frac{z+1}{3}$, $\quad p_2 \ldots \frac{x-3}{-2}=\frac{y+1}{3}=\frac{z-2}{4}$

**Rj:** općenito: (ako nisu paralelni (i ne sijeku se)) $T_1$ = točka na $p_1$, $T_2$ = točka na $p_2$.

$\Pi$ = ravnina paralelna s $p_1$ koja sadrži $p_2$




$q_1$ = pravac kroz $T_2$ paralelan s $p_1$.

volumen paralelepipeda $=|(\vec{T_2T_1},\vec{s_1},\vec{s_2})|=$

$=|\vec{s_1}\times\vec{s_2}|\cdot v$, $\quad v=d(p_1,p_2)$

$d(p_1,p_2)=d(p_1,\Pi)$

$$\Rightarrow d(p_1,p_2)=\frac{|(\vec{r_1}-\vec{r_2},\vec{s_1},\vec{s_2})|}{|\vec{s_1}\times\vec{s_2}|}$$

Kod nas $\vec{s_1}=(2,-4,3)$, $\vec{s_2}=(-2,3,4)$ $\quad \vec{s_1}\times\vec{s_2}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\2&-4&3\\-2&3&4\end{vmatrix}=(-25,-14,-2)$

$|\vec{s_1}\times\vec{s_2}|=5\sqrt{33}$ $\quad \vec{r_1}=(-5,3,-1)$, $\vec{r_2}=(3,-1,2)$

$\vec{r_1}-\vec{r_2}=(-8,4,-3)$

$$(\vec{r}_1-\vec{r}_2, \vec{A}_1, \vec{A}_2) = (\vec{A}_1, \vec{A}_2, \vec{r}_1-\vec{r}_2) = (\vec{A}_1 \times \vec{A}_2)\cdot(\vec{r}_1-\vec{r}_2) =$$

$$= (-25,-14,-2)\cdot(-8,4,-3) = 200-56+6 = 150$$

$$\Rightarrow \quad d(p_1,p_2) = \frac{150}{5\sqrt{33}} = \frac{30\sqrt{33}}{33} = \frac{10}{11}\sqrt{33} \;//$$

D7

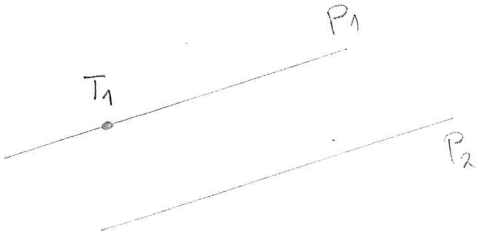
**Zad 23.:** Odredite udaljenost dvaju paralelnih pravaca

~~DA~~!

$$p_1 \ldots \frac{x-1}{2} = \frac{y+1}{2} = \frac{z-3}{-1}$$

$$p_2 \ldots \frac{x}{-2} = \frac{y-1}{-2} = \frac{z+4}{1}$$

**Rj:** Pravac $p_1$ ima vektor smjera $(2,2,-1)$, a pravac $p_2$ vektor smjera $(-2,-2,1)$. Kako su im vektori smjera kolinearni, pravci $p_1$ i $p_2$ su zaista paralelni.




$d(p_1,p_2) = d(T_1,p_2)$

Sve točke na $p_1$ su jednako udaljene od $p_2$.

Uzmimo npr. $T_1=(1,-1,3)$, $T_2=(0,1,-4)$. Prema formuli iz zadatka 21 vrijedi $d(T_1,p_2) = |(\vec{r}_1-\vec{r}_2)\times\vec{A}| / |\vec{A}| =$

$$= \frac{\left|\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 7 \\ 2 & 2 & -1 \end{vmatrix}\right|}{|(2,2,-1)|} = \frac{|(-12,15,6)|}{\sqrt{9}} = |(4,5,2)| = \sqrt{16+25+4} = 3\sqrt{5} \;//$$

**Zad 24.:** Dana je ravnina $\Pi \ldots x+y-z+1=0$ i pravac

NE! SAMO c)

$$p \ldots \frac{x-1}{0} = \frac{y}{2} = \frac{z+1}{1}.$$

a) Odredite njihovo sjecište i $\measuredangle(\Pi, p)$

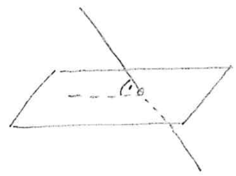
b) Odredite jednadžbu ravnine koja sadrži pravac $p$, a okomita je na ravninu $\Pi$.

c) Odredite jednadžbu projekcije pravca $p$ na ravninu $\Pi$.

Rj: a) $p \ldots \begin{cases} x = 1 \\ y = 2t \\ z = t-1 \end{cases}, \quad t \in \mathbb{R}$

$1 + 2t - (t-1) + 1 = 0 \Rightarrow t + 3 = 0 \Rightarrow \boxed{t = -3} \Rightarrow$ sjecište je točka $(1, -6, -4)$.

$\sphericalangle(\pi, p) = \frac{\pi}{2} - \underbrace{\sphericalangle(\vec{n}, \vec{s})}_{\in [0, \frac{\pi}{2}]}$ $\qquad \vec{n} = (1, 1, -1), \quad \vec{s} = (0, 2, 1)$



$\cos \sphericalangle(\vec{n}, \vec{s}) = \frac{\vec{n} \cdot \vec{s}}{|\vec{n}| \cdot |\vec{s}|} = \frac{2-1}{\sqrt{3} \cdot \sqrt{5}} = \frac{1}{\sqrt{15}}$

$\sphericalangle(\pi, p) = \frac{\pi}{2} - \arccos \frac{1}{\sqrt{15}}$

b) Vektor normale tražene ravnine $\tilde{\pi}$ mora biti okomit na $(0,2,1)$ (jer je $p \subseteq \tilde{\pi}$ i $\vec{n} \perp \tilde{\pi}$) i na $(1,1,-1)$ (jer je $\pi \perp \tilde{\pi}$).

Stoga je $\tilde{n} = (1,1,-1) \times (0,2,1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 0 & 2 & 1 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 1 & 1 \\ 0 & 2 \end{matrix} = \vec{i} + 2\vec{k} - \vec{j} + 2\vec{i} =$

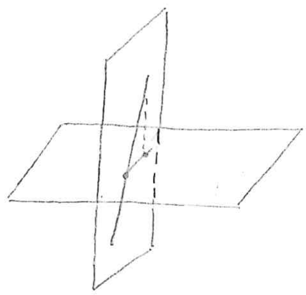
$= 3\vec{i} - \vec{j} + 2\vec{k}$

$\Rightarrow \vec{\tilde{n}} = (3, -1, 2) \Rightarrow \tilde{\pi} \ldots \ 3x - y + 2z + D = 0$

$(1, 0, -1) \in p \subseteq \tilde{\pi} \Rightarrow 3 - 2 + D = 0 \Rightarrow \boxed{D = -1}$

$\Rightarrow \tilde{\pi} \ldots \ 3x - y + 2z - 1 = 0$

c)



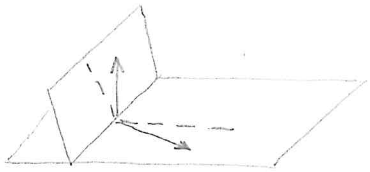
Kako $\tilde{\pi}$ sadrži $p$ i $\tilde{\pi} \perp \pi$, to je ortogonalna projekcija pravca $p$ na ravninu $\pi$ upravo presjek ravnina $\tilde{\pi} \cap \pi$:

$$\begin{cases} x + y - z + 1 = 0 \\ 3x - y + 2z - 1 = 0 \end{cases}$$

**Zad 25.:** Odredite jednadžbu ravnine koja sadrži $Ox$ i koja s ravninom $y = x$ zatvara kut od $60°$.

DA!

rj:



Kut između dviju ravnina je kut između normala tih dviju ravnina.

Ravnina $y=x$ ima vektor normale $(-1,1,0)$. Neka je $\vec{n}=(a,b,c)$ vektor normale tražene ravnine, t.d. $|\vec{n}|=1$, $\sphericalangle(\vec{n},\vec{n}_1)=60^\circ$

$\vec{n}\cdot\vec{n}_1=-a+b$ , $\vec{n}\cdot\vec{n}_1=|\vec{n}_1|\cdot|\vec{n}|\cdot\cos 60^\circ=\sqrt{2}\cdot 1\cdot\frac{1}{2}=\frac{1}{\sqrt{2}}$

$\Rightarrow \boxed{-a+b=\frac{1}{\sqrt{2}}}$

Ravnina sadrži os $x$ $\Rightarrow$ vektor smjera osi $x$ $(1,0,0)$ je okomit na vektor normale $\vec{n}$ pa je $\boxed{a=0}$, a onda je $\boxed{b=\frac{1}{\sqrt{2}}}$.

$c$ određujemo iz uvjeta $|\vec{n}|=1 \Rightarrow \sqrt{\frac{1}{2}+c^2}=1 \Rightarrow c^2=\frac{1}{2} \Rightarrow c=\pm\frac{1}{\sqrt{2}}$

$n=\left(0,\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}(0,1,\pm 1)$

Tražena ravnina sadrži $x$-os pa specijalno sadrži i ishodište.

Rješenje su dvije ravnine: $\pi_1 \ldots y+z=0$

$\pi_2 \ldots y-z=0$

**Zad 26.:** Odredite jednadžbe simetrala kutova koje zatvaraju pravci:

DA!

$$p\ldots \frac{x+5}{-3}=\frac{y-14}{6}=\frac{z+3}{2} \quad i \quad q\ldots \frac{x-3}{2}=\frac{y+1}{-3}=\frac{z+1}{6}$$

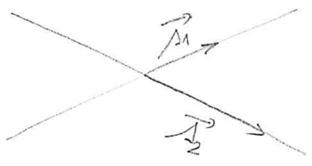
rj: Točka presjeka pravaca $p$ i $q$:

$$p\ldots\begin{cases} x=-5-3t\\ y=14+6t\\ z=-3+2t\end{cases} \qquad q\ldots\begin{cases} x=3+2s\\ y=-1-3s\\ z=-1+6s\end{cases}$$

$$\begin{aligned}
-5-3t&=3+2s &&\Rightarrow 3t+2s=-8 &&\Rightarrow 3t+2s=-8\\
14+6t&=-1-3s &&\Rightarrow 6t+3s=-15\ /:3 &&\Rightarrow 2t+s=-5\\
-3+2t&=-1+6s &&\Rightarrow 2t-6s=2\ /:2 && \phantom{\Rightarrow}\ t-3s=1
\end{aligned}\Bigg\} \Rightarrow \boxed{s=-1}, \boxed{t=-2}$$

$\boxed{p \cap q = (1, 2, -7)}$

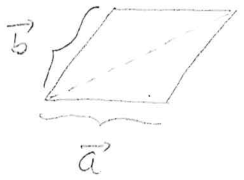
Vektor smjera pravca p je $\vec{s}_1 = (-3, 6, 2)$, a pravca q $\vec{s}_2 = (2, -3, 6)$




Vektor smjera simetrale tih dvaju pravaca je $\dfrac{\vec{s}_1}{|\vec{s}_1|} + \dfrac{\vec{s}_2}{|\vec{s}_2|} = \dfrac{(-3,6,2)}{\sqrt{49}} + \dfrac{(2,-3,6)}{\sqrt{49}} =$

$= \frac{1}{7}(-1, 3, 8)$, a simetrala

prolazi točkom $(1, 2, -7)$ pa je jednadžba simetrale kuta




simetrala kuta $\measuredangle(\vec{a}, \vec{b})$ za $|\vec{a}| = |\vec{b}|$ je $\vec{a} + \vec{b}$

$$\frac{x-1}{-1} = \frac{y-2}{3} = \frac{z+7}{8}$$

**Kut između dvaju pravaca** = kut između njihovih vektora smjera.

DA!

Ako je riječ o pravcima u ravnini, onda imamo:

$p_1 \ldots \quad A_1 x + B_1 y + C_1 = 0 \quad$ ili $\quad y = k_1 x + l_1 \qquad \vec{n}_1 = (A_1, B_1)$

$p_2 \ldots \quad A_2 x + B_2 y + C_2 = 0 \quad$ ili $\quad y = k_2 x + l_2 \qquad \vec{s}_1 = (-B_1, A_1)$

$B_1 y = -A_1 x - C_1$

$\dfrac{y}{A_1} = \dfrac{x + \frac{C_1}{A_1}}{-B_1}$

$$\cos \measuredangle(p_1, p_2) = \frac{|\vec{s}_1 \cdot \vec{s}_2|}{|\vec{s}_1| \cdot |\vec{s}_2|} = \frac{|(-B_1, A_1) \cdot (-B_2, A_2)|}{\sqrt{A_1^2 + B_1^2} \cdot \sqrt{A_2^2 + B_2^2}} =$$

$$= \frac{|A_1 A_2 + B_1 B_2|}{\sqrt{A_1^2 + B_1^2} \cdot \sqrt{A_2^2 + B_2}} = \cos \text{ kuta između vektora normale}$$

$$\cos \measuredangle(p_1, p_2) = \frac{|\vec{s}_1 \cdot \vec{s}_2|}{|\vec{s}_1| \cdot |\vec{s}_2|} = \frac{|(1, k_1) \cdot (1, k_2)|}{\sqrt{1 + k_1^2}\ \sqrt{1 + k_2^2}} = \frac{|1 + k_1 k_2|}{\sqrt{1 + k_1^2} \cdot \sqrt{1 + k_2^2}}$$

$$\operatorname{tg}^2\varphi = \frac{\sin^2\varphi}{\cos^2\varphi} = \frac{1-\cos^2\varphi}{\cos^2\varphi} = \frac{1}{\cos^2\varphi} - 1$$

$$\operatorname{tg}^2\varphi = \frac{(1+k_1^2)(1+k_2^2)}{(1+k_1k_2)^2} - 1 = \frac{\cancel{1}+k_1^2+k_2^2+\cancel{k_1^2k_2^2}-\cancel{1}-2k_1k_2-\cancel{k_1^2k_2^2}}{(1+k_1k_2)^2} =$$

$$= \frac{(k_1-k_2)^2}{(1+k_1k_2)^2} \Rightarrow \operatorname{tg}\varphi = \frac{|k_1-k_2|}{|1+k_1k_2|}$$

DZ

Zad 27.: Dan je pravac $p \ldots 3x-2y+5=0$ i točke $A=(1,4)$,
~~NE~~! $B=(5,2)$. Odredite $d(B,p)$, $\measuredangle(AB,p)$ i $AB\cap p$.

Rj: pravac $AB \ldots$ $y-4 = \frac{2-4}{5-1}(x-1)$, $y-4 = -\frac{1}{2}x + \frac{1}{2}$

$$y = -\frac{1}{2}x + \frac{9}{2} \qquad k_1 = -\frac{1}{2}$$

$$p \ldots \quad y = \frac{3}{2}x + \frac{5}{2} \qquad k_2 = \frac{3}{2}$$

$$\operatorname{tg}\varphi = \left|\frac{\frac{3}{2}+\frac{1}{2}}{1-\frac{3}{4}}\right| = \frac{2}{\frac{1}{4}} = 8 \qquad \varphi = \operatorname{arctg} 8$$

$$d(B,p) = \frac{|3\cdot 5 - 2\cdot 2 + 5|}{\sqrt{3^2+2^2}} = \frac{16}{\sqrt{13}}.$$

$AB\cap p$ je rješenje sustava $\begin{cases} 3x-2y+5=0 \\ y = -\frac{1}{2}x+\frac{9}{2} \end{cases}$

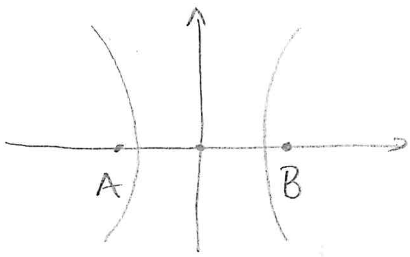
$$\Rightarrow 3x-(-x+9)+5=0 \Rightarrow 4x-4=0 \Rightarrow \boxed{x=1} \Rightarrow \boxed{y=4}$$

$$\boxed{AB\cap p = (1,4)}$$

Elipsa, hiperbola i parabola (1.)

1. Neka su A i B točke u ravnini i konstanta $2a < |AB|$. Odredite geometrijsko mjesto svih točaka X t.d. $|AX|-|BX| = 2a$

Rj:




Ishodište stavimo u polovište dužine $\overline{AB}$.

$A=(-t,0)$

$B=(t,0)$

$2a<2t \quad a<t$

$2a \cancel{>} 2t \Rightarrow a>t$

$t^2=a^2+b^2$

$|AX|-|BX|=a$

$\sqrt{(x+t)^2+y^2}-\sqrt{(x-t)^2+y^2}=2a$

$\sqrt{(x+t)^2+y^2}=2a+\sqrt{(x-t)^2+y^2} \quad /^2$

$\cancel{x^2}+2xt+\cancel{t^2}+\cancel{y^2}=4a^2+4a\sqrt{(x-t)^2+y^2}+\cancel{x^2}-2xt+\cancel{t^2}+\cancel{y^2}$

$4xt=4a^2+4a\sqrt{(x-t)^2+y^2}$

$xt-a^2=a\sqrt{(x-t)^2+y^2} \quad /^2$

$x^2t^2+a^4-2a^2xt=a^2(x^2-2xt+t^2+y^2)$

$x^2t^2+a^4-\cancel{2a^2xt}=a^2x^2-\cancel{2a^2xt}+a^2t^2+a^2y^2$

$x^2t^2+a^4-a^2t^2-a^2x^2-a^2y^2=0$

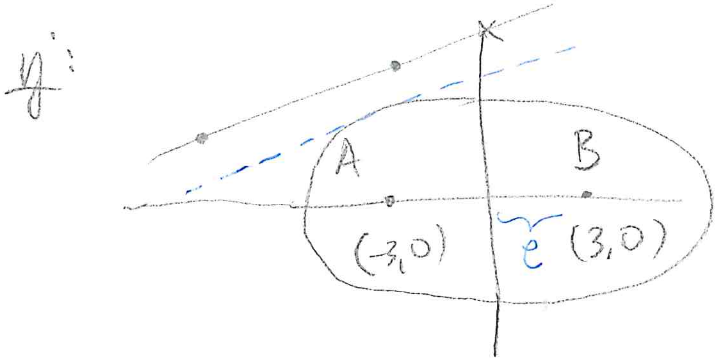
$x^2(\underbrace{t^2-a^2}_{=b^2})-a^2y^2=a^2t^2-a^4 \quad /: \begin{array}{l} a^2t^2-a^4 \\ =a^2(t^2-a^2) \end{array}$

$$\frac{x^2}{\frac{a^2b^2}{b^2}}-\frac{y^2}{\frac{a^2b^2}{a^2}}=1 \Rightarrow \boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1}$$

hiperbola

A i B su žarišta, a realna poluos, b imaginarna

(2.) Neka je $A(-3,0)$ i $B(3,0)$. Među svim točkama $X$ t.d. $|AX|+|BX|=10$, odredite onu koja je najbliža pravcu koji prolazi točkama $(-3,4)$ i $(-8,1)$.




$|AX|+|BX|=10$ $\qquad 2a=10$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\qquad a=5$

ekscentricitet

$e=3$

$\frac{x^2}{25}+\frac{y^2}{16}=1$ $\qquad e^2=a^2-b^2$

$9=25-b^2$

$16x^2+25y^2=400$

$$k=\frac{1-4}{-8-(-3)}=\frac{-3}{-5}=\frac{3}{5}$$

Tražimo tangentu elipse s tim koeficijentom smjera.

Uvjet dodira: $\quad y=kx+l$

$b^2x^2+a^2y^2=a^2b^2$

$b^2x^2+a^2(k^2x^2+2klx+l^2)=a^2b^2$

$(b^2+a^2k^2)x^2+2kla^2x+a^2l^2-a^2b^2=0$

$D=0 \quad 4k^2l^2a^4-4a^2(l^2-b^2)(b^2+a^2k^2)=0$

$\Rightarrow k^2l^2a^2-(l^2-b^2)(b^2+a^2k^2)=0$

$\Rightarrow \cancel{k^2l^2a^2}-l^2b^2-\cancel{l^2a^2k^2}+b^4+b^2a^2k^2=0$

$b^4+b^2a^2k^2-l^2b^2=0 \quad /:b^2$

$\boxed{b^2+a^2k^2=l^2}$

$16+\cancel{25}\cdot\frac{9}{\cancel{25}}=l^2 \Rightarrow l^2=25$

$\Rightarrow l=\pm5$

Treba nam gornja tangenta $y=\frac{3}{5}x+5$

$$\left.\begin{array}{l} 16x^2+25y^2=400 \\ y=\frac{3}{5}x+5 \end{array}\right\}$$ rj: je $x=-3$, $y=\frac{16}{5}$

$\Rightarrow$ Tražena točka je $\left(-3,\frac{16}{5}\right)$

## Konike kao krivulje drugog reda

① Navedite tri načina za određivanje jednadžbe kružnice koja prolazi kroz točke $(-2,-1)$, $(0,3)$, $(6,5)$.

rj: I° $(x-x_0)^2+(y-y_0)^2=r^2$

Uvrstimo sve tri točke i dobijemo sustav s 3 jednadžbe, 3 nepoznanice.

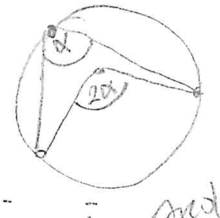
$$\left.\begin{array}{l} 5+4x_0+2y_0+x_0^2+y_0^2=r^2 \\ 9 \quad\quad -6y_0+x_0^2+y_0^2=r^2 \\ 61-12x_0-10y_0+x_0^2+y_0^2=r^2 \end{array}\right\} \Rightarrow x_0, y_0$$

Uvrstimo u prvu i dobijemo $r$.

$$(-2-x_0)^2+(-1-y_0)^2=(0-x_0)^2+(3-y_0)^2=(6-x_0)^2+(5-y_0)^2$$

$\Rightarrow x_0, y_0 \Rightarrow$ uvrstimo i dobijemo $r$.

II° Računamo simetrale stranica trokuta ABC i tražimo njihov presjek.




III° koristimo obodni i središnji kut

(2.) Odredite kut za koji treba rotirati koordinatni sustav tako da zapis jednadžbe

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0$$

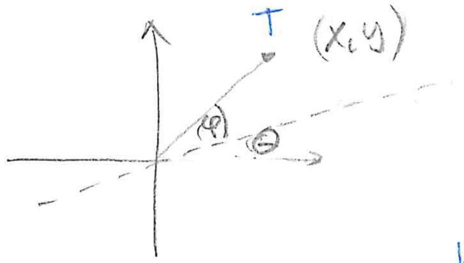
u novim koordinatama ima koeficijent 0 uz $xy$.

stare, nove, stare

Rj: $x=\hat{x}\cos\theta-\hat{y}\sin\theta$, $\quad y=\hat{x}\sin\theta+\hat{y}\cos\theta$

izvod

točke T koordinate u novom sustavu




$$(\hat{x},\hat{y})=r(\cos\varphi,\sin\varphi)$$

$$(x,y)=r(\cos(\varphi+\theta),\sin(\varphi+\theta))$$

koordinate u novom sustavu

$$=r(\cos\varphi\cos\theta-\sin\varphi\sin\theta,\ \sin\varphi\cos\theta+\sin\theta\cos\varphi)$$

$$=((r\cos\varphi)\cdot\cos\theta-r\sin\varphi\sin\theta,\ r\sin\varphi\cos\theta+r\cos\varphi\sin\theta)$$

$$=(\hat{x}\cos\theta-\hat{y}\sin\theta,\ \hat{y}\cos\theta+\hat{x}\sin\theta)$$

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$$A(\hat{x}\cos\theta-\hat{y}\sin\theta)^2+2B(\hat{x}\cos\theta-\hat{y}\sin\theta)(\hat{x}\sin\theta+\hat{y}\cos\theta)$$
$$+C(\hat{x}\sin\theta+\hat{y}\cos\theta)^2+2D(\hat{x}\cos\theta-\hat{y}\sin\theta)+2E(\hat{x}\sin\theta+\hat{y}\cos\theta)$$
$$+F=0$$

Uz $\hat{x}\hat{y}$ mora biti 0.

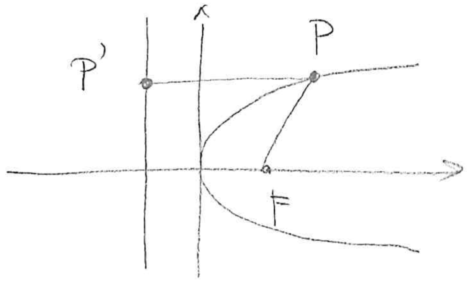
Sređivanjem $2B\left((\cos\theta)^2-(\sin\theta)^2\right)=2(A-C)\cos\theta\sin\theta$

$$\Rightarrow \boxed{\operatorname{ctg}(2\theta)=\frac{A-C}{2B}}$$

# Optička svojstva konika i primjene

1. (Optičko svojstvo parabole) Neka je P točka na paraboli sa žarištem F i neka je P' ortogonalna projekcija točke P na ravnalicu parabole. Dokažite da je tangenta na parabolu u točki P simetrala kuta $\measuredangle P'PF$.

R:




$P=(x,y)$

$F=(\frac{p}{2},0)$

$P'=(-\frac{p}{2},y)$

$|PP'|^2=|x+\frac{p}{2}|^2$

$|PF|^2=(x-\frac{p}{2})^2+y^2$

$\cancel{x^2}-px+\cancel{\frac{p^2}{4}}+y^2=\cancel{x^2}+px+\cancel{\frac{p^2}{4}}$

$y^2=2px$

$|PP'|=|PF| \Rightarrow \Delta P'PF$ je jednakokračan

Dovoljno je dokazati da je tangenta u točki P okomita na P'F.

UVJET DODIRA: $y^2=2px$

$(kx+l)^2-2px=0$

$k^2x^2+(2kl-2p)x+l^2=0$

$0=D=4(kl-p)^2-4k^2l^2=4p^2-8klp$

$\Rightarrow p-2kl=0 \Rightarrow p=2kl \Rightarrow y=kx+\frac{p}{2k}$

$(x,y)\in$ parabola $\Rightarrow x=\frac{p}{2k^2}$, $y=\frac{p}{k}$

P'F ima k-smjera $\frac{y}{-p}=-\frac{1}{k} \Rightarrow$ P'F je okomit na tangentu.

① Odredite jednadžbu kružnice koja prolazi točkom $T=(1,0)$ i dira pravce $x+y-2=0$ i $x+y+3=0$.

rj: $y=kx+l$ ... p

$(x-a)^2+(y-b)^2=r^2$ ... k

p dira k $\iff$ kvadratna jednadžba $(x-a)^2+(kx+l-b)^2=r^2$ ima samo jedno rješenje.

$$x^2-2xa+a^2+k^2x^2+2kx(l-b)+(l-b)^2-r^2=0$$

$\Delta=0$ $\quad (2k(l-b)-2a)^2-4(1+k^2)((l-b)^2-r^2+a^2)=0 \;/:4$

$$\cancel{k^2(l-b)^2}-2ak(l-b)+\cancel{a^2}-(l-b)^2+r^2-\cancel{a^2}-\cancel{k^2(l-b)^2}+k^2r^2-k^2a^2=0$$

$$r^2(1+k^2)=(l-b)^2+k^2a^2+2ak(l-b)$$

$$r^2(1+k^2)=(l-b+ak)^2$$

$$\boxed{r^2(1+k^2)=(ka-b+l)^2}$$

$p_1 \ldots y=-x+2$

$p_2 \ldots y=-x-3$

$$r^2(1+1^2)=(-a-b+2)^2$$
$$=(-a-b-3)^2$$

$$\boxed{2r^2=(a+b-2)^2=(a+b+3)^2}$$

$-4(a+b)+4=6(a+b)+9$

$-5=10(a+b)$

$a+b=-\frac{1}{2} \Rightarrow 2r^2=\left(-\frac{5}{2}\right)^2$

$r^2=\frac{1}{2}\cdot\frac{25}{4}=\frac{25}{8}$

$T\in k \Rightarrow (1-a)^2+b^2=r^2$

$(1-a)^2+\left(-\frac{1}{2}-a\right)^2=\frac{25}{8} \;/\cdot 4$

$(2-2a)^2+(2a+1)^2=\frac{25}{2}$

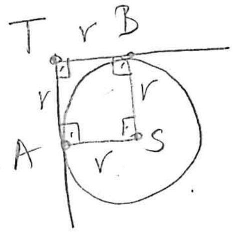
$4-8a+4a^2+4a^2+4a+1=\frac{25}{2}$

$8a^2-4a=\frac{25}{2}-5=\frac{15}{2}$

$8a^2-4a-\frac{15}{2}=0 \quad \Rightarrow a,b$

② Odredite skup svih točaka ravnine iz kojih se kružnica $x^2+y^2=r^2$ vidi pod pravim kutem.

Rj:

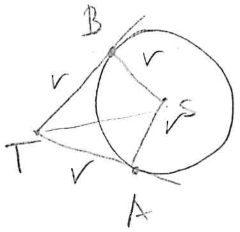



$|TS|^2=2r^2 \Rightarrow |TS|=r\sqrt{2}$

$\Rightarrow T\in k(S, r\sqrt{2})$

Obratno, za $T'\in k(S, r\sqrt{2})$, povučemo tangente $TA$ i $TB$ na kružnicu.

$|TA|=|TB|=\sqrt{|TS|^2-r^2}=\sqrt{2r^2-r^2}=\sqrt{r^2}=r$



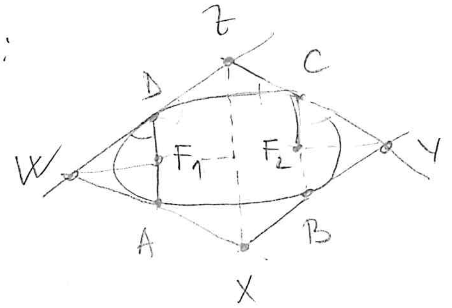

$\Rightarrow \angle BTS=\angle BSA=45^\circ$

$\angle ATS=45^\circ$

$\Rightarrow \angle ATB=90^\circ$.

③ U onim točkama elipse $8x^2+9y^2-72=0$ koje imaju apscise jednake apscisama njezinih fokusa povučene su tangente. Dokažite da te tangente određuju romb. Izračunajte površinu romba.

Rj:




$8x^2+9y^2=72 \;/:72$

$\frac{x^2}{9}+\frac{y^2}{8}=1$

$e^2=a^2-b^2=9-8=1$

$e=1$

Uvjet dodira pravca $y=kx+l$ i elipse:

$b^2+a^2k^2=l^2$

$8+9k^2=l^2$

$F_1=(-1,0)$, $F_2=(1,0)$

$A=(-1, \frac{8}{3})$

$D=(-1, -\frac{8}{3})$

$B=(1, -\frac{8}{3})$

$C=(1, \frac{8}{3})$

Jednadžba tangente na elipsu $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ u točki $(x_1, y_1)$ je

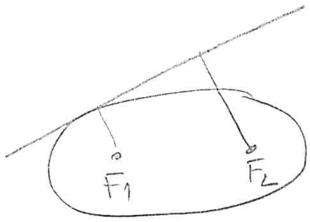
$b^2x_1x+a^2y_1y=a^2b^2$

Odredimo tangente u točkama $A, B, C, D$. Odredimo njihove presjeke $X, Y, Z, W$. $XYZW$ je paralelogram.

$WY \perp XZ$.

4.) Dokažite da je produkt udaljenosti bilo koje tangente elipse $b^2x^2+a^2y^2=a^2b^2$ od njenih fokusa konstantan i jednak kvadratu male poluosi.

rj:



$F_1=(-e,0)$ $\qquad e^2=a^2-b^2$
$F_2=(e,0)$

$t\ldots\ b^2x_1x+a^2y_1y=a^2b^2$

Računamo $$\frac{|b^2x_1\cdot(-e)-a^2b^2|}{\sqrt{(b^2x_1)^2+(a^2y_1)^2}}\cdot\frac{|b^2x_1\cdot e-a^2b^2|}{\sqrt{(b^2x_1)^2+(a^2y_1)^2}}=$$

$$=\frac{b^2|-x_1e-a^2|\cdot b^2|x_1e-a^2|}{b^4x_1^2+a^4y_1^2}=$$

$$=b^4\cdot\frac{|(x_1e)^2-a^4|}{b^4x_1^2+a^4y_1^2}=b^4\cdot\frac{|x_1^2(a^2-b^2)-a^4|}{b^4x_1^2+a^2\cdot(a^2b^2-b^2x_1^2)}=$$

$$=b^4\frac{|a^4-x_1^2(a^2-b^2)|}{b^4x_1^2+a^4b^2-a^2b^2x_1^2}=b^2\frac{|a^4-x_1^2(a^2-b^2)|}{a^4-x_1^2(a^2-b^2)}=b^2$$

~~5.) Dokažite da se tangente parabole povučene krajevima fokalne tetive sijeku na direktrisi.~~

5.) Kako glasi jednadžba hiperbole kojoj je pravac $3x-4y-10=0$ tangenta, a fokusi su joj u točkama $F_1=(-5,0)$, $F_2=(5,0)$?

Uvjet dodira pravca i hiperbole: $a^2k^2-b^2=l^2$

$3x-4y-10=0 \qquad 4y=3x-10$
$\qquad y=\frac{3}{4}x-\frac{5}{2}$

$a^2\cdot\frac{9}{16}-b^2=\frac{25}{4}\Rightarrow b^2=\frac{9}{16}a^2-\frac{25}{4}$

$e=\sqrt{a^2+b^2}\Rightarrow 5=\sqrt{a^2+b^2}\Rightarrow a^2+b^2=25$

$$25 - a^2 = \frac{9}{16}a^2 - \frac{25}{4} \Rightarrow \frac{125}{4} = \frac{25}{4}a^2 \ /: \frac{25}{4}$$

$$5 = a^2 \Rightarrow b^2 = 20$$

$$\frac{x^2}{5} - \frac{y^2}{20} = 1$$

$y = kx + l$ dira $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \iff$

$\frac{x^2}{a^2} - \frac{(kx+l)^2}{b^2} = 1$ ima samo jedno rj.

$$b^2x^2 - a^2(kx+l)^2 - a^2b^2 = 0$$

$$b^2x^2 - a^2(k^2x^2 + 2klx + l^2) - a^2b^2 = 0$$

$$\Delta = (+a^2 \cdot 2kl)^2 - 4(b^2 - a^2k^2)(-a^2l^2 - a^2b^2) = 0 \quad /: 4a^2$$

$$a^2 \cdot k^2 l^2 + (b^2 - a^2k^2)(l^2 + b^2) = 0$$

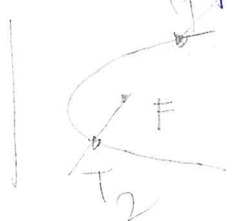
$$\cancel{a^2k^2l^2} + b^2l^2 + b^4 - \cancel{a^2k^2l^2} - a^2k^2b^2 = 0 \quad /: b^2$$

$$l^2 + b^2 = a^2k^2$$

$$\iff a^2k^2 - b^2 = l^2$$

Zad: Fokusom parabole $y^2 = 2x$ povučen je pravac paralelan s pravcem $2x - y = 0$. Dokažite da se tangente povučene u sjecištima tog pravca s parabolom sijeku na direktrisi parabole.

Rj:




$$y^2 = 2 \cdot 1 \cdot x$$

$$F = \left(\frac{1}{2}, 0\right)$$

$$d \ldots y = -\frac{1}{2}$$

$p \ldots y = 2x + l$

$F \in p \Rightarrow 0 = 2 \cdot \frac{1}{2} + l \Rightarrow l = -1 \qquad p \ldots y = 2x - 1$

$$(2x-1)^2=2x$$

$$4x^2-4x+1=2x$$

$$4x^2-6x+1=0$$

$$x_{1,2}=\frac{6\pm\sqrt{36-16}}{8}=\frac{6\pm\sqrt{20}}{8}=\frac{3\pm\sqrt5}{4} \qquad y_{1,2}=\frac{3\pm\sqrt5}{2}-1=\frac{1\pm\sqrt5}{2}$$

Jednadžba tangente parabole $y^2=2px$ u $T(x_0,y_0)$ je

$$y_0\cdot y=p\cdot(x+x_0)$$

$$t_{1,2}\ldots \quad \frac{1\pm\sqrt5}{2}y=x+\frac{3\pm\sqrt5}{4}$$

$$t_1\cap t_2=0 \qquad y_0=\frac{x_0+\frac{3\pm\sqrt5}{4}}{\frac{1\pm\sqrt5}{2}}$$

$$\frac{x_0+\frac{3+\sqrt5}{4}}{\frac{1+\sqrt5}{2}}=\frac{x_0+\frac{3-\sqrt5}{4}}{\frac{1-\sqrt5}{2}} \quad \Big/\cdot\ \frac{1+\sqrt5}{2}\cdot\frac{1-\sqrt5}{2}$$

$$\left(x_0+\frac{3+\sqrt5}{4}\right)\frac{1-\sqrt5}{2}=\left(x_0+\frac{3-\sqrt5}{4}\right)\cdot\frac{1+\sqrt5}{2}$$

$$\frac{-2-2\sqrt5}{8}-\frac{-2+2\sqrt5}{8}=x_0\cdot\sqrt5$$

$$-\frac{\sqrt5}{2}=x_0\cdot\sqrt5 \Rightarrow x_0=-\frac12$$

$\Rightarrow$ $t_1$ i $t_2$ se sijeku na ravnalici

Zad: Neka je $p$ bilo koji pravac kroz fokus $F$ parabole $y^2=2px$, a $S_1$ i $S_2$ sjecišta pravca $p$ s parabolom. Dokažite da je broj $\frac{1}{d(F,S_1)}+\frac{1}{d(F,S_2)}$ konstantan za tu parabolu.

$F = \left(\frac{p}{2}, 0\right)$

p ... prava kroz F

$y = kx + l$

$0 = k \cdot \frac{p}{2} + l \Rightarrow l = -\frac{kp}{2}$

p ... $y = kx - \frac{kp}{2} = k\left(x - \frac{p}{2}\right)$

p ∩ parabola $\quad k^2\left(x - \frac{p}{2}\right)^2 = 2px$

$$x^2 - px + \frac{p^2}{4} = \frac{2p}{k^2}x$$

$$x^2 - p\left(1 + \frac{2}{k^2}\right)x + \frac{p^2}{4} = 0$$

$$x_{1,2} = \frac{p\left(1 + \frac{2}{k^2}\right) \pm \sqrt{p^2\left(1 + \frac{2}{k^2}\right)^2 - p^2}}{2}$$

$$= \frac{p\left(1 + \frac{2}{k^2}\right) \pm \sqrt{p^2\left(+\frac{4}{k^2} + \frac{4}{k^4}\right)}}{2}$$

$$y_{1,2} = k\left(\frac{p}{k^2} \pm p\sqrt{\frac{1}{k^4} + \frac{1}{k^2}}\right)$$

$$\frac{1}{d(F, S_1)} + \frac{1}{d(F, S_2)} = \frac{1}{\sqrt{\left(x_1 - \frac{p}{2}\right)^2 + y_1^2}} + \frac{1}{\sqrt{\left(x_2 - \frac{p}{2}\right)^2 + y_2^2}} =$$

$$= \frac{1}{\sqrt{\left(x_1 - \frac{p}{2}\right)^2 + k^2\left(x_1 - \frac{p}{2}\right)^2}} + \frac{1}{\sqrt{\left(x_2 - \frac{p}{2}\right)^2 + k^2\left(x_2 - \frac{p}{2}\right)^2}} =$$

$$= \frac{1}{\sqrt{1 + k^2}} \cdot \left(\frac{1}{\left|x_1 - \frac{p}{2}\right|} + \frac{1}{\left|x_2 - \frac{p}{2}\right|}\right)$$

$$= \frac{1}{\sqrt{1 + k^2}} \cdot \left(\frac{1}{\frac{p}{k^2}\left|1 + \sqrt{1 + k^2}\right|} + \frac{1}{\frac{p}{k^2}\left|1 - \sqrt{1 + k^2}\right|}\right) =$$

$$= \frac{1}{\frac{\sqrt{1+k^2}}{k^2} p} \cdot \frac{|1-\sqrt{1+k^2}| + |1+\sqrt{1+k^2}|}{|1-(1+k^2)|} =$$

$$= \frac{1}{p \cdot \sqrt{1+k^2}} \left(\sqrt{1+k^2} \cancel{-1} + \sqrt{1+k^2} \cancel{+1}\right) = \frac{2}{p}$$

Zad: Odredite jednadžbu elipse kojoj su fokusi $(\pm 3, 0)$, a pravac $x-y-5=0$ tangenta. Iz kojih točaka na $y$-osi se ta elipsa vidi pod pravim kutem?

$t \ldots y = x-5$

$e^2 = a^2 - b^2 \Rightarrow a^2 - b^2 = 9$

uvjet dodira $a^2k^2 + b^2 = l^2$

$$a^2 + b^2 = 25$$
$$a^2 - b^2 = 9$$

$2a^2 = 34$

$\boxed{a^2 = 17 \Rightarrow b^2 = 8}$ $\qquad \frac{x^2}{17} + \frac{y^2}{8} = 1$

$T(0, y_T)$

Tangenta na elipsu u točki $(x_0, y_0)$ $\quad b^2 x_0 x + a^2 y_0 y = a^2 b^2$

$T(0, y_T) \in t \Rightarrow \left.\begin{aligned} y_1 y_T &= 8 \\ y_2 y_T &= 8 \end{aligned}\right\} \Rightarrow y_1 = y_2$

Kut između tangenti $\quad 8x_1 x + 17 y_1 y = 8 \cdot 17$

$8x_2 x + 17 y_2 y = 8 \cdot 17$

$\cos 90^\circ = 0 \Rightarrow |A_1A_2 + B_1B_2| = 0 \Rightarrow 8^2 x_1 x_2 + 17^2 y_1 y_2 = 0$

$8^2 x_1 x_2 + 17^2 y_1^2 = 0$

$y_1 = y_2$

$\Rightarrow \frac{y_1^2}{8} = \frac{y_2^2}{8}$

$\Rightarrow \frac{x_1^2}{17} = \frac{x_2^2}{17}$

$\Rightarrow x_1 = \pm x_2$

Za $x_1 = x_2$:

$8^2 x_1^2 + 17^2 y_1^2 = 0$

$\Rightarrow x_1 = y_1 = 0$ ↯

(0,0) nije na elipsi

Za $x_1 = -x_2$:

$-8^2 x_1^2 + 17^2 y_1^2 = 0$

$-8^2 x_1^2 + 17^2 \cdot 8 \cdot \left(1 - \frac{x_1^2}{17}\right) = 0$

$-8^2 x_1^2 + 17^2 \cdot 8 - 17 \cdot 8 x_1^2 = 0$

$17^2 \cdot 8 = 8(17+8) x_1^2 \quad /:8$

$17^2 = 25 x_1^2$

$\frac{17^2}{25} = x_1^2 \Rightarrow \quad 17^2 y_1^2 = 8^2 \cdot \frac{17^2}{25}$

$y_1^2 = \frac{8^2}{5^2} \Rightarrow y_1 = \pm \frac{8}{5}$

$\Rightarrow y_T = \pm \frac{8}{\frac{8}{5}} = \pm 5$

Iz točaka $(0,5)$ i $(0,-5)$.